REPRESENTATION THEORY

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Chapter 1

Representations

1.1 Basic Definitions

Groups are intended to describe symmetries of geometric and other mathematical objects. Representations are symmetries of some of the most basic objects in geometry and algebra, namely vector spaces.

Representations have three different aspects — geometric, numerical and algebraic — and manifest themselves in corresponding form. We begin with the numerical form.

In a general context we write groups G in multiplicative form. The group structure (multiplication) is then a map $G \times G \to G, (g,h) \mapsto g \cdot h = gh$, the unit element is e or 1, and g^{-1} is the inverse of g.

An n-dimensional ***matrix representation*** of the group G over the field K is a homomorphism $\varphi\colon G \to GL_n(K)$ into the general linear group $GL_n(K)$ of invertible (n,n)-matrices with entries in K. Two such representations φ, ψ are said to be ***conjugate*** if there exists a matrix $A \in GL_n(K)$ such that the relation $A\varphi(g)A^{-1} = \psi(g)$ holds for all $g \in G$. The representation φ is called ***faithful*** if φ is injective. If $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}$, we talk about complex, real, and rational representations.

The group $GL_1(K)$ will be identified with the multiplicative group of non-zero field elements $K^* = K \smallsetminus \{0\}$. In this case we are just considering homomorphisms $G \to K^*$.

Next we come to the geometric form of a representation as a symmetry group of a vector space. The field K will be fixed.

A ***representation*** of G on the K-vector space V, a KG-representation for short, is a map

$$\rho\colon G \times V \to V, \quad (g,v) \mapsto \rho(g,v) = g \cdot v = gv$$

with the properties:

(1) $g(hv) = (gh)v, ev = v$ for all $g, h \in G$ and $v \in V$.
(2) The ***left translation*** $l_g \colon V \to V, v \mapsto gv$ is a K-linear map for each $g \in G$.

We call V the ***representation space*** Its dimension as a vector space is the ***dimension*** $\dim V$ of the representation (sometimes called the ***degree*** of the representation). The rules (1) are equivalent to $l_g \circ l_h = l_{gh}$ and $l_e = \mathrm{id}_V$. They express the fact that ρ is a group action – see the next section. From $l_g l_{g^{-1}} = l_{gg^{-1}} = l_e = \mathrm{id}$ we see that l_g is a linear isomorphism with inverse $l_{g^{-1}}$.

Occasionally it will be convenient to define a representation as a map

$$V \times G \to V, \quad (v, g) \mapsto vg$$

with the properties $v(hg) = (vh)g$ and $ve = v$, and K-linear right translations $r_g \colon v \mapsto vg$. These will be called ***right*** representations as opposed to ***left*** representations defined above. The map $r_g \colon v \mapsto vg$ is then the ***right translation*** by g. Note that now $r_g \circ r_h = r_{hg}$ (contravariance). If V is a right representation, then $(g, v) \mapsto vg^{-1}$ defines a left representation. We work with left representations if nothing else is specified.

One can also use both notions simultaneously. A (G, H)***-representation*** is a vector space V with the structure of a left G-representation and a right H-representation, and these structures are assumed to commute $(gv)h = g(vh)$.

A ***morphism*** $f \colon V \to W$ between KG-representations is a K-linear map f which is G***-equivariant***, i.e., which satisfies $f(gv) = gf(v)$ for $g \in G$ and $v \in V$. Morphisms are also called ***intertwining operators***. A bijective morphism is an ***isomorphism***. The vector space of all morphisms $V \to W$ is denoted $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$. Finite-dimensional KG-representations and their morphisms form a category KG-Rep.

Let V be an n-dimensional representation of G over K. Let B be a basis of V and denote by $\varphi^B(g) \in GL_n(K)$ the matrix of l_g with respect to B. Then $g \mapsto \varphi^B(g)$ is a matrix representation of G. Conversely, from a matrix representation we get in this manner a representation.

(1.1.1) Proposition. *Let V, W be representations of G, and B, C bases of V, W. Then V, W are isomorphic if and only if the corresponding matrix representations φ^B, φ^C are conjugate.*

Proof. Let $f \colon V \to W$ be an isomorphism and A its matrix with respect to B, C. The equivariance $f \circ l_g = l_g \circ f$ then translates into $A\varphi^B(g) = \varphi^C(g)A$; and conversely. □

Conjugate 1-dimensional representations are equal. Therefore the isomorphism classes of 1-dimensional representations correspond bijectively to homomorphisms $G \to K^*$. The aim of representation theory is not to determine matrix representations. But certain concepts are easier to explain with the help of matrices.

Let V be a representation of G. A ***sub-representation*** of V is a subspace U which is G-invariant, i.e., $gu \in U$ for $g \in G$ and $u \in U$. A non-zero representation V is called ***irreducible*** if it has no sub-representations other than $\{0\}$ and V. A representation which is not irreducible is called ***reducible***.

(1.1.2) [Schur's Lemma] Let V and W be irreducible representations of G.
(1) A morphism $f\colon V \to W$ is either zero or an isomorphism.
(2) If K is algebraically closed then a morphism $f\colon V \to V$ is a scalar multiple of the identity, $f = \lambda \cdot \mathrm{id}$.

Proof. (1) Kernel and image of f are sub-representations. If $f \neq 0$, then the kernel is different from V hence equal to $\{0\}$ and the image is different from $\{0\}$ hence equal to W.

(2) Algebraically closed means: Non-constant polynomials have a root. Therefore f has an eigenvalue $\lambda \in K$ (root of the characteristic polynomial). Let $V(\lambda)$ be the eigenspace and $v \in V(\lambda)$. Then $f(gv) = gf(v) = g(\lambda v) = \lambda gv$. Therefore $gv \in V(\lambda)$, and $V(\lambda)$ is a sub-representation. By irreducibility, $V = V(\lambda)$. □

(1.1.3) Proposition. *An irreducible representation of an abelian group G over an algebraically closed field is one-dimensional.*

Proof. Since G is abelian, the l_g are morphisms and, by (1.1.2), multiples of the identity. Hence each subspace is a sub-representation. □

(1.1.4) Example. Let S_n be the symmetric group of permutations of $\{1, \ldots, n\}$. We obtain a right(!) representation of S_n on K^n by permutation of coordinates

$$K^n \times S_n \to K^n, \quad ((x_1, \ldots, x_n), \sigma) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This representation is not irreducible if $n > 1$. It has the sub-representations $T_n = \{(x_i) \mid \sum_{i=1}^n x_i = 0\}$ and $D = \{(x, \ldots, x) \mid x \in K\}$. ◇

Schur's lemma can be expressed in a different way. Recall that an ***algebra*** A over K consists of a K-vector space together with a K-bilinear map $A \times A \to A, (a, b) \mapsto ab$ (the multiplication of the algebra). The algebra is called associative (commutative), if the multiplication is associative (commutative). An associative algebra with unit element is therefore a ring with the additional property that the multiplication is bilinear with respect to the scalar multiplication in the vector space. In a ***division algebra*** (also called ***skew field***) any non-zero element has a multiplicative inverse. A typical example of an associative algebra is the endomorphism algebra $\mathrm{Hom}_G(V, V)$ of a representation V; multiplication is the composition of endomorphisms. Other examples are the algebra $M_n(K)$ of (n, n)-matrices with entries in K and the polynomial algebra $K[x]$. The next proposition is a reformulation of Schur's lemma.

(1.1.5) Proposition. *Let V be an irreducible G-representation. Then the endomorphism algebra $A = \mathrm{Hom}_G(V, V)$ is a division algebra. If K is algebraically closed, then $K \to A$, $\lambda \mapsto \lambda \cdot \mathrm{id}$ is an isomorphism of K-algebras.* □

Let V be an irreducible G-representation over $\mathbb{R}$. A finite-dimensional division algebra over $\mathbb{R}$ is one of the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$. We call V of ***real, complex, quaternionic type*** according to the type of its endomorphism algebra.

The third form of a representation — namely a module over the group algebra — will be introduced later.

(1.1.6) [Cyclic groups] The cyclic group of order n is the additive group $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n$ of integers modulo n. We also use a formal multiplicative notation for this group $C_n = \langle\, a \mid a^n = 1 \,\rangle$; this means: a is a generator and the n-th power is the unit element.

Homomorphisms $\alpha\colon C_n \to H$ into another group H correspond bijectively to elements $h \in H$ such that $h^n = 1$, via $a \mapsto \alpha(a)$. Hence there are n different 1-dimensional representations over the complex numbers $\mathbb{C}$, given by $a \mapsto \exp(2\pi i t/n)$, $0 \le t < n$.

The rotation matrices $D(\alpha)$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy $D(\alpha)D(\beta) = D(\alpha + \beta)$ and $BD(\alpha)B^{-1} = D(-\alpha)$. We obtain a 2-dimensional real representation $\varphi_t\colon a \mapsto D(2\pi t/n)$. The representations φ_t and $\varphi_{-t} = \varphi_{n-t}$ are conjugate. ◇

(1.1.7) [Dihedral groups] Groups can be presented in terms of generators and relations. We do not enter the theory of such presentations but consider an example. Let

$$D_{2n} = \langle\, a, b \mid a^n = 1 = b^2,\ bab^{-1} = a^{-1} \,\rangle.$$

This means: The group is generated by two elements a and b, and these generators satisfy the specified relations. The ***universal property*** of this presentation is: The homomorphisms $\alpha\colon D_{2n} \to H$ into any other group H correspond bijectively to pairs $(A = \alpha(a), B = \alpha(b))$ in H such that $A^n = 1 = B^2, BAB^{-1} = A^{-1}$.

Thus 1-dimensional representations over $\mathbb{C}$ correspond to complex numbers A, B such that $A^n = B^2 = A^2 = 1$. If n is odd there are two pairs $(1, \pm 1)$; if n is even there are four pairs $(\pm 1, \pm 1)$.

A 2-dimensional representation on the $\mathbb{R}$-vector space $\mathbb{C}$ is specified by $a \cdot z = \lambda z$, $b \cdot z = \overline{z}$ where $\lambda^n = 1$. Complex conjugation shows that the representations which correspond to λ and $\overline{\lambda}$ are isomorphic. Denote the representation obtained from $\lambda = \exp(2\pi i t/n)$ by V_t.

The group D_{2n} has order $2n$ and is called the ***dihedral group*** of this order. From a geometric viewpoint, D_{2n} is the orthogonal symmetry group of the regular n-gon in the plane. A faithful matrix representation $D_{2n} \to O(2)$ is obtained by choosing $\lambda = \exp(2\pi i/n)$. The powers of a correspond to rotations, the elements $a^t b$ to reflections. ◇

(1.1.8) Example. The real representations $\varphi_t, 1 \le t < n/2$, of C_n in (1.1.6) are irreducible. A nontrivial sub-representation would be one-dimensional and spanned by an eigenvector of $\varphi_t(a)$.

If we consider φ_t as a complex representation, then it is no longer irreducible, since eigenvectors exist. In terms of matrices

$$PD(\alpha)P^{-1} = \begin{pmatrix} \exp(i\alpha) & 0 \\ 0 & \exp(-i\alpha) \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The representations $V_t, 1 \le t < n/2$ of D_{2n} in (1.1.7) are irreducible, since they are already irreducible as representations of C_n. But this time they remain irreducible when considered as complex representations. The reason is, that PBP^{-1} does not preserve the eigenspaces. ◇

Problems

1. The dihedral group D_{2n} has the presentation $\langle\, s, t \mid s^2 = t^2 = (st)^n = 1 \,\rangle$.
2. Recall the notion of a semi-direct product of groups and show that D_{2n} is the semi-direct product of C_n by C_2.
3. Let $Q_{4n} = \langle\, a, b \mid a^n = b^2, bab^{-1} = a^{-1} \,\rangle$, $n \ge 2$. Deduce from the relation $b^4 = a^{2n} = 1$. Show that Q_{4n} is a group of order $4n$. Show that $a \mapsto \exp(\pi i/n), b \mapsto j$ induces an isomorphism of Q_{4n} with a subgroup of the multiplicative group of the quaternions. The group Q_{4n} is called a ***quaternion group***. Show that Q_{4n} has also the presentation $\langle\, s, t \mid s^2 = t^2 = (st)^n \,\rangle$. Construct a two-dimensional faithful irreducible (matrix) representation over $\mathbb{C}$.
4. Let A_4 be the alternating group of order 12 (even permutations in S_4). Show: A_4 has 3 elements of order 2, 8 elements of order 3. Show that A_4 is the semi-direct product of $C_2 \times C_2$ by C_3. Show $A_4 = \langle\, a, b \mid a^3 = b^3 = (ab)^2 = 1 \,\rangle$; show that $s = ab$ and $t = ba$ are commuting elements of order 2 which generate a normal subgroup. Show that the matrices

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

define a representation on $\mathbb{R}^3$ as a symmetry group of a regular tetrahedron.
5. A representation of C_2 on V amounts to specifying an involution $T\colon V \to V$, i.e., a linear map T with $T^2 = \mathrm{id}$. If the field K has characteristic different from 2 show that V is the direct sum of the ± 1-eigenspaces of T. (Consider the operators $\frac{1}{2}(1+T)$ and $\frac{1}{2}(1-T)$.)

1.2 Group Actions and Permutation Representations

In this section we collect basic terminology about group actions. We use group actions to construct the important class of permutation representations.

Let G be a multiplicative group with unit element e. A ***left action*** of a group G on a set X is a map

$$\rho\colon G \times X \to X, \quad (g,x) \mapsto \rho(g,x) = g \cdot x = gx$$

with the properties $g(hx) = (gh)x$ and $ex = x$ for $g, h \in G$ and $x \in X$. The pair (X, ρ) is called a ***(left) G-set***. Each $g \in G$ yields the ***left translation*** $l_g\colon X \to X$, $x \mapsto gx$ by g. It is a bijection with inverse the left translation by g^{-1}. An action is called ***effective***, if l_g for $g \neq e$ is never the identity. We also use ***(right) actions*** $X \times G \to X$, $(x,g) \mapsto xg$. They satisfy $(xh)g = x(hg)$ and $xe = x$. Usually we work with left G-actions.

A subset A of a G-set X is called ***G-stable*** or ***G-invariant***, if $g \in G$ and $a \in A$ implies $ga \in A$.

Recall that we defined a representation as a group action on a vector space with the additional property that the left translations are linear maps. We now use group actions to construct representations.

Let S be a finite (left) G-set and denote by KS the vector space with K-basis S. Thus elements in KS are linear combinations $\sum_{s \in S} \lambda_s s$ with $\lambda_s \in K$. The left action of G on S is extended linearly to KS

$$g \cdot (\textstyle\sum_{s \in S} \lambda_s s) = \sum_{s \in S} \lambda_s (g \cdot s) = \sum_{x \in S} \lambda_{g^{-1}x} x.$$

The resulting representation is called the ***permutation representation*** of S. An important example is obtained from the group $G = S$ with left action by group multiplication. The associated permutation representation is the ***left regular representation*** of the finite group G. Right multiplication leads to the right regular representation.

Let X be a G-set. Then $R = \{(x, gx) \mid x \in X, g \in G\}$ is an equivalence relation on X. Let X/G denote the set of equivalence classes. The class of x is $Gx = \{gx \mid g \in G\}$ and called the ***orbit*** through x. We call X/G the ***orbit set*** or (***orbit space*** of the G-set X. An action is called ***transitive***, if it consists of a single orbit. For systematic reasons it would be better to denote the orbit set of a left action by $G\backslash X$. If right and left actions occur, we use both notations.

A group acts on itself by conjugation $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$. Elements are ***conjugate*** if they are in the same orbit. The orbits are called ***conjugation classes***. A function on G is called a ***class function***. if it is constant on conjugacy classes.

Let H be a subgroup of G. We have the set G/H of right cosets gH with left G-action by left translation

$$G \times G/H \to G/H, \quad (k, gH) \mapsto kgH.$$

G-sets of this form are called ***homogeneous*** G-sets. Similarly, we have the set $H\backslash G$ of left cosets Hg with an action by right translation.

We write $H \leq G$, if H is a subgroup of G, and $H < G$, if it is a proper subgroup. On the set $\mathrm{Sub}(G)$ of subgroups of G the relation $\leq$ is a partial order.

The group G acts on $\mathrm{Sub}(G)$ by conjugation $(g,H) \mapsto gHg^{-1} = {}^gH$. The orbit through H consists of the subgroups H of G which are ***conjugate*** to H. We write $K \sim L$ or $K \sim_G L$, if there exists $g \in G$ such that $gKg^{-1} = L$. We denote by (H) the conjugacy class of H. Let $\mathrm{Con}(G)$ be the set of conjugacy classes of subgroups of G. We say H is ***subconjugate*** to K in G, if H is conjugate in G to a subgroup of K. We denote this fact by $(H) \leq (K)$; and by $(H) < (K)$, if equality is excluded.

The ***stabilizer*** or ***isotropy group*** of $x \in X$ is the subgroup $G_x = \{g \in G \mid gx = x\}$. We have $G_{gx} = gG_xg^{-1}$. An action is called ***free***, if all isotropy groups are trivial. The set of isotropy groups of X is denoted $\mathrm{Iso}(X)$.

A ***family*** $\mathcal{F}$ ***of subgroups*** is a subset of $\mathrm{Sub}(G)$ which consists of complete conjugacy classes. If $\mathcal{F}$ and $\mathcal{G}$ are families, we write $\mathcal{F} \circ \mathcal{G}$ for the family of intersections $\{K \cap L \mid K \in \mathcal{F}, L \in \mathcal{G}\}$. We call $\mathcal{F}$ ***multiplicative***, if $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$, and $\mathcal{G}$ is called $\mathcal{F}$***-modular***, if $\mathcal{F} \circ \mathcal{G} \subset \mathcal{G}$. A family is called ***closed***, if it contains with a group all supergroups, and it is called ***open***, if it contains with a group all subgroups. Let $(\mathcal{F})$ denote the set of conjugacy classes of $\mathcal{F}$. Suppose $\mathrm{Iso}(X) \subset \mathcal{F}$, then we call X an $\mathcal{F}$-set. We denote by $X(\mathcal{F})$ the subset of points in X with isotropy groups in $\mathcal{F}$.

A G***-map*** $f \colon X \to Y$ between G-sets, also called a G***-equivariant map***, is a map which satisfies $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Left G-sets and G-equivariant maps form the category G-SET. By passage to orbits, a G-map $f \colon X \to Y$ induces $f/G \colon X/G \to Y/G$. The category G-SET of G-sets and G-maps has products: If $(X_j \mid j \in J)$ is a family of G-sets, then the Cartesian product $\prod_{j \in J} X_j$ with so-called ***diagonal action*** $g(x_j) = (gx_j)$ is a product in this category.

(1.2.1) Proposition. *Let C be a transitive G-set and $c \in C$. Then $G/G_c \to C$, $gG_c \mapsto gc$ is a well-defined isomorphism of G-sets (a simple algebraic verifi-*

cation). The orbits of a G-set are transitive. Therefore each G-set is isomorphic to a disjoint sum of homogeneous G-sets. □

For a G-set X and a subgroup H of G we use the following notations

$$\begin{aligned} X_H &= \{x \in X \mid G_x = H\}, \\ X_{(H)} &= \{x \in X \mid (G_x) = (H)\} \\ X^H &= \{x \in X \mid hx = x,\ h \in H\}, \\ X^{>H} &= X^H \smallsetminus X_H \\ X^{(H)} &= GX^H = \{x \in X \mid (H) \leq (G_x)\}, \\ X^{>(H)} &= X^{(H)} \smallsetminus X_{(H)}. \end{aligned}$$

We call X^H the ***H-fixed point set*** of X. If $f\colon X \to Y$ is a G-map, then $f(X^H) \subset Y^H$. The left translation $l_g\colon X \to X$ induces a bijection $X^H \to X^K$, $K = gHg^{-1}$. The subset $X_{(H)}$ is G-stable; it is called the ***(H)-orbit bundle*** of X.

(1.2.2) Example. The permutation representation KS is always reducible ($|S| > 1$). The fixed point set is a non-zero proper sub-representation. The dimension of $(KS)^G$ is $|S/G|$; a basis of $(KS)^G$ consists of the $x_C = \sum_{s \in C} s$ where C runs through the orbits of S. ◇

Suppose X is a right and Y a left H-set. Then $X \times_H Y$ denotes the quotient of $X \times Y$ with respect to the equivalence relation $(xh, y) \sim (x, hy)$, $h \in H$. This is the orbit set of the action $(h, (x, y)) \mapsto (xh^{-1}, hy)$ of H on $X \times Y$.

Let G and H be groups. A ***(G,H)-set*** X is a set X together with a left G-action and a right H-action which commute $(gx)h = g(xh)$. If we form $X \times_H Y$, then this set carries an induced G-action $g \cdot (x, y) = (gx, y)$. If $f\colon Y_1 \to Y_2$ is an H-map, then we obtain an induced G-map $X \times_H f\colon X \times_H Y_1 \to X \times_H Y_2$. This construction yields a functor $\rho(X)\colon H\text{-}\mathrm{SET} \to G\text{-}\mathrm{SET}$.

We apply this construction to the (G,H)-set $G = X$ with action by left G-translation and right H-translation for $H \leq G$. The resulting functor is called ***induction functor***

$$\mathrm{ind}_H^G\colon H\text{-}\mathrm{SET} \to G\text{-}\mathrm{SET}\,.$$

It is left adjoint to the ***restriction functor***

$$\mathrm{res}_H^G\colon G\text{-}\mathrm{SET} \to H\text{-}\mathrm{SET},$$

given by considering a G-set as an H-set. The adjointness means that there is a natural bijection

$$\mathrm{Hom}_G(\mathrm{ind}_H^G X, Y) \cong \mathrm{Hom}_H(X, \mathrm{res}_H^G Y).$$

It assigns to an H-map $f\colon X \to Y$ the G-map $G \times_H X \to Y$, $(g,x) \mapsto gf(x)$.

One can obtain interesting group theoretic results by counting orbits and fixed points. We give some examples.

(1.2.3) Proposition. *Let P be a p-group and X a finite P-set; then $|X| \equiv |X^P| \bmod p$. Let C be cyclic of order p^t and $D \leq C$ the unique subgroup of order p; then $|X| \equiv |X^D| \bmod p^t$.*

Proof. Each orbit in $X \smallsetminus X^P$ has cardinality divisible by p. Each orbit in $X \smallsetminus X^D$ has cardinality p^t. □

(1.2.4) Proposition. *Let $P \neq 1$ be a p-group. Then P has a non-trivial center $Z(P) = \{x \in P \mid \forall y \in P,\, xy = yx\}$.*

Proof. Let P act on itself by conjugation $(x,y) \mapsto xyx^{-1}$. The fixed point set is the center. Since $1 \in Z(P)$, we see from (1.2.3) that $|Z(P)|$ is non-zero and divisible by p. □

(1.2.5) Proposition. *Let P be a p-group. There exists a chain of normal subgroups $P_i \lhd P$*

$$1 = P_0 \lhd P_1 \lhd \ldots \lhd P_r = P$$

such that $|P_i/P_{i-1}| = p$.

Proof. Induct on $|P|$. Since subgroups of the center are normal, there exists by (1.2.4) a normal subgroup P_1 of order p. Apply the induction hypothesis to the factor group P/P_1 and lift a normal series to P. □

(1.2.6) Proposition. *Let K be a field of characteristic p and V a KP-representation for a p-group P. Then $V^P \neq \{0\}$.*

Proof. Let P have order p with generator x. Then $l_x\colon V \to V$ has eigenvalues a root of $X^p - 1 = (X-1)^p$. Hence 1 is the only eigenvalue. In the general case choose a normal subgroup $Q \lhd P$ and observe that V^Q is a $K(P/Q)$-representation. □

A group A is called ***elementary abelian of rank*** n if it is isomorphic to the n-fold product $(C_p)^n$ of cyclic groups C_p of prime order p. We can view this group as n-dimensional vector space over the prime field $\mathbb{F}_p$.

(1.2.7) Proposition. *Let the p-group P act on the elementary abelian p-group A of rank n by automorphisms. Then there exists a chain $1 = A_0 < \ldots < A_n = A$ of subgroups which are P-invariant and $|A_i/A_{i-1}| = p$.* □

(1.2.8) [Counting lemma] Let G be a finite group, M a finite G-set, and $\langle\, g \,\rangle$ the cyclic subgroup generated by $g \in G$. Then $|G| \cdot |M/G| = \sum_{g \in G} |M^{\langle\, g\,\rangle}|$.

Proof. Let $X = \{(g,x) \in G \times M \mid gx = x\}$. Consider the maps

$$p\colon X \to G,\ (g,x) \mapsto g, \qquad q\colon X \to M/G,\ (g,x) \mapsto Gx.$$

Since $p^{-1}(g) = |\{g\} \times M^{\langle g \rangle}|$, the right hand side is the sum of the cardinalities of the fibres of p. Since $|q^{-1}(Gx)| = |G_x||Gx| = |G|$, the left hand side is the sum of the cardinalities of the fibres of q. □

Problems

1. Let X and Y be G-sets. The set $\mathrm{Hom}(X,Y)$ of all maps $X \to Y$ carries a left G-action $(g \cdot f)(x) = gf(g^{-1}x)$. The G-fixed point set $\mathrm{Hom}(X,Y)^G$ is the subset $\mathrm{Hom}_G(X,Y)$ of G-maps $X \to Y$.
2. Let X be a G-set and $H \le G$. Then $G \times_H X \to G/H \times X$, $(g,x) \mapsto (gH, gx)$ is a bijection of G-sets. If Y is a further H-set, then we have an isomorphism of G-sets $G \times_H (X \times Y) \cong X \times (G \times_H Y)$.
3. Determine the conjugacy classes of D_{2n} and Q_{4n}.
4. The orbits of $G/K \times G/L$ correspond bijectively to the double cosets $K\backslash G/L$; the maps

$$\begin{aligned} G\backslash(G/K \times G/L) &\to K\backslash G/L, && G \cdot (uK, vL) \mapsto Ku^{-1}vL \\ K\backslash G/L &\to G\backslash(G/K \times G/L), && v \mapsto G \cdot (eK, vL) \end{aligned}$$

are inverse bijections.

1.3 The Orbit Category

The full subcategory of G-SET with object the homogeneous G-sets is called the ***orbit category*** $\mathrm{Or}(G)$ of G.

(1.3.1) Proposition. *Let H and K be subgroups of G.*
(1) *There exists a G-map $G/H \to G/K$ if and only if (H) is subconjugate to (K).*
(2) *Each G-map $G/H \to G/K$ has the form $R_a\colon gH \mapsto gaK$ for an $a \in G$ such that $a^{-1}Ha \subset K$.*
(3) *$R_a = R_b$ if and only if $a^{-1}b \in K$.*
(4) *G/H and G/K are G-isomorphic if and only if H and K are conjugate in G.*

Proof. Let $f\colon G/H \to G/K$ be equivariant and suppose $f(eH) = aK$. By equivariance, we have for all $h \in H$ the equalities $aK = f(eH) = f(hH) = hf(eH) = haK$ and hence $a^{-1}Ha \subset K$. The other assertions are easily verified. □

We denote by $N_G H = NH = \{n \in G \mid nHn^{-1} = H\}$ the ***normalizer*** of H in G and by $W_G(H) = WH$ the associated quotient group NH/H (***Weyl-group***.) Suppose G is finite. Then $n^{-1}Hn \subset H$ implies $n^{-1}Hn = H$. Hence each endomorphism of G/H is an automorphism. A G-map $f\colon G/H \to G/H$ has the form $gH \mapsto gn_f H$ for a uniquely determined coset $n_f \in WH$. The assignment $f \mapsto n_f^{-1}$ is an isomorphism $\mathrm{Aut}_G(G/H) \cong WH$.

(1.3.2) Proposition. *The right action of the automorphism group*

$$G/H \times WH \to G/H, \quad (gH, nH) \mapsto gnH$$

is free. Hence for each $K \leq G$ the set G/H^K carries a free WH-action and the cardinality $|G/H^K|$ is divisible by $|WH|$. We have $G/H^H = WH$. □

(1.3.3) Example. The assignment

$$\Psi_L\colon \mathrm{Hom}_G(G/L, X) \to X^L, \quad \alpha \mapsto \alpha(eL)$$

is a bijection. The inverse sends $x \in X^L$ to $gL \mapsto gx$. We have

$$G/L^K = \{sL \mid s^{-1}Ks \leq L\}.$$

Given $sL \in G/L^K$ then $R_s\colon G/K \to G/L, gK \mapsto gsL$ is the associated morphism. The diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(G/L, X) & \xrightarrow{\Psi_L} & X^L \\ \downarrow{\scriptstyle R_s^*} & & \downarrow{\scriptstyle l_s} \\ \mathrm{Hom}_G(G/K, X) & \xrightarrow{\Psi_K} & X^K \end{array}$$

is commutative. We view the Ψ_L as a natural isomorphism from the Hom-functor $\mathrm{Hom}_G(-, X)$ to the ***fixed point functor***. The left translation by $n \in NK$ maps X^K into itself. In this way, X^K becomes a WK-set. ◇

Let G be a finite group. The fixed point set G/L^K is the set $\{sL \mid s^{-1}Ks \leq L\}$. Let $A \leq L$ be G-conjugate to K. Consider the subset

$$G/L^K(A) = \{tL \mid t^{-1}Kt \sim_L A\}.$$

The set G/L^K has a left $N_G K$-action $(n, sL) \mapsto nsL$. The subsets $G/L^K(A)$ are $N_G K$-invariant.

(1.3.4) Proposition. *Suppose $s^{-1}Ks = A$. The assignment*

$$N_G(A)/N_L(A) \to G/L^K(A), \quad nN_L(A) \mapsto snL$$

is a bijection.

Proof. Since $n^{-1}s^{-1}Ksn = n^{-1}An = A \le L$ the element snL is contained in G/L^K. The map is well-defined, because $N_L(A) \le A$. If $snL = smL$, then $m^{-1}n \in L \cap N_G(A) = N_L(A)$, and we see that the map is injective.

Suppose $A \sim_L t^{-1}Kt \le L$. Then there exists $l \in L$ such that $t^{-1}Kt = l^{-1}Al$, hence $s^{-1}tl^{-1} \in N_G(A)$, and $tL = snL$. We see that the map is surjective. □

We can rewrite this result in terms of the $N_G(K)$-action on G/L^K. The subset $G/L^K(A)$ is an $N_G(K)$-orbit; and the isotropy group at sL is $sN_L(A)s^{-1}$. The fixed point set G/L^K is the disjoint union of the $G/L^K(A)$ where (A) runs over the L-conjugacy classes of the subgroups $A \le L$ which are G-conjugate to K.

Since the homogeneous G-sets correspond to the subgroups of G we can consider a modified orbit category: The objects are the subgroups of G and the morphisms $K \to L$ are the G-maps $G/K \to G/L$. Since we are working with left actions we denote this category by ${}_\bullet\mathrm{Or}(G)$. There is a similar category $\mathrm{Or}_\bullet(G)$ where the morphisms $K \to L$ are the G-maps $K\backslash G \to L\backslash G$. If we assign to $R_s\colon G/K \to G/L$ the map $L_{s^{-1}}\colon K\backslash G \to L\backslash G, Kg \mapsto Ls^{-1}g$, then we obtain an isomorphism ${}_\bullet\mathrm{Or}(G) \to \mathrm{Or}_\bullet(G)$.

The ***transport category*** $\mathrm{Tra}(G)$ of G has as object set the subgroups of G, and the morphism set $\mathrm{Tra}(K,L)$ consists of the triples (K,L,s) with $s \in G$ and $sKs^{-1} \subset L$. We denote $\mathrm{Tra}(K,L)$ also as $\{s \in G \mid sKs^{-1} \le L\}$ and pretend that the morphism sets are disjoint. Composition is defined by multiplication of group elements. In this context we work with the orbit category $\mathrm{Or}_\bullet(G)$ of right homogeneous G-sets. We have a functor

$$q\colon \mathrm{Tra}(G) \to \mathrm{Or}_\bullet(G).$$

It is the identity on objects and sends (K,L,s) to $l_s\colon K\backslash G \to L\backslash G,\ Kg \mapsto Lsg$. The endomorphism sets in both categories are groups

$$\mathrm{Tra}(K,K) = NK, \qquad \mathrm{Or}_\bullet(K,K) = WK.$$

Via composition, $\mathrm{Tra}(K,L)$ carries a left action of $NL = \mathrm{Tra}(L,L)$ and a right action of $NK = \mathrm{Tra}(K,K)$. These actions commute. Similarly for the category $\mathrm{Or}_\bullet(G)$. The functor q is surjective on Hom-sets and induces a bijection

$$L\backslash\mathrm{Tra}(K,L) \cong \mathrm{Or}_\bullet(K,L).$$

Let

$$(K,L)_* = \{A \mid A \le L, A \sim_G K\} \tag{1.1}$$
$$(K,L)^* = \{B \mid K \le B, B \sim_G L\}. \tag{1.2}$$

(These sets can be empty.) We have bijections

$$\mathrm{Tra}(K,L)/NK \cong (K,L)_*, \quad s \cdot NK \mapsto sKs^{-1},$$

$$NL\backslash\mathrm{Tra}(K,L) \cong (K,L)^*, \quad NL \cdot s \mapsto s^{-1}Ls.$$

They imply the counting identities

$$|(K,L)_*| \cdot |NK| = |(K,L)^*| \cdot |NL| = |G/L^K| \cdot |L|. \tag{1.3}$$

The integers $\zeta^*(K,L) = |(K,L)^*|$ and $\zeta_*(K,L) = |(K,L)_*|$ depend only on the conjugacy classes of K and L. The $\mathrm{Con}(G) \times \mathrm{Con}(G)$-matrices ζ^* and ζ_* have the property that their entries at $(K),(L)$ are zero if $(K) \not\leq (L)$, and the diagonal entries are 1. They are therefore invertible over $\mathbb{Z}$, and their respective inverses μ^* and μ_* have similar properties. In order to see this, ones solves the equation

$$\textstyle\sum_{(A)} \zeta^*(K,A)\mu^*(A,L) = \delta_{(K),(L)} \tag{1.4}$$

inductively for $\mu^*(K,L)$; the induction is over $|\{(A) \mid (K) \leq (A) \leq (L)\}|$. The matrices μ^* and μ_* are called the ***Möbius-matrices*** of $\mathrm{Con}(G)$. Let N denote the diagonal matrix with entry $|NK|$ at $(K),(K)$. Then ((1.3)) says

$$\zeta_* N = N\zeta^*, \qquad N\mu_* = \mu^* N.$$

Let G be abelian. Then

$$\zeta^*(K,L) = \zeta_*(K,L) = \zeta(K,L) = 1, \quad \text{for } (K) \leq (L).$$

Hence also $\mu^* = \mu_* = \mu$ in this case.

(1.3.5) Proposition. *Let $G \cong (\mathbb{Z}/p)^d$ be elementary abelian. Then $\mu(1,G) = (-1)^d p^{d(d-1)/2}$.*

Proof. The direct proof from the definition is a classical q-identity. For an indeterminate q we define the quantum number

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1},$$

and the q-binomial coefficient

$$[n]_q! = [1]_q[2]_q \cdots [n_q], \quad \binom{n}{a}_q = \frac{[n]_q!}{[a]_q![n-a]_q!}.$$

With a further indeterminate z the following generalized binomial identity holds

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}_q q^{j(j-1)/2} z^j = \prod_{k=0}^{n-1} (1 - q^k z). \tag{1.5}$$

A proof can be given by induction over n, as in the case of the classical binomial identity. If q is a prime, then $\binom{n}{j}_q$ is the number of j-dimensional subspaces of $\mathbb{F}_q^n$. For $z = 1$ the identity ((1.5)) yields inductively the values of the μ-function as claimed. □

There are two more quotient categories of the transport category. The ***homomorphism category*** $Sc(G)$ has as morphism set the homomorphisms $K \to L$ which are of the form $k \mapsto gkg^{-1}$ for some $g \in G$. Two elements of G define the same homomorphism if they differ by an element of the centralizer ZK of K in G. We can therefore identify the morphism set $\mathrm{Sc}(K, L)$ with $\mathrm{Tra}(K, L)/ZK$.

Finally we can combine the orbit category and the homomorphism category. In the category $\mathrm{Sci}(G)$ we consider homomorphisms $K \to L$ up to inner automorphisms of L; thus the morphism set $\mathrm{Sci}(K, L)$ can be identified with the double coset $L\backslash\mathrm{Tra}(K, L)/ZK$.

Problems

1. S be a G-set and $K \le G$. We have a free left WK-action on S_K via left translation. The inclusion $S_K \subset S_{(K)}$ induces a bijection $S_K/WK \cong S_{(K)}/G$. The map

$$G/K \times_{WK} S_K \to S_{(K)}, \quad (gK, x) \mapsto gx$$

is a bijection of G-sets.
2. Suppose $D \le G$ is cyclic. Then $|D||G/D^A| = |N_G A|$ if $(A) \le (D)$. Hence $\zeta_*(A, D) = 1$ for $(A) \le (D)$. If $\mu\colon \mathbb{N} \to \mathbb{Z}$ denotes the classical Möbius-function, then $\mu_*(A, D) = \mu(|D/A|)$ and $\mu^*(A, D) = NA/ND\mu(|D/A|)$. (The function μ is defined inductively by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ in the case that $n > 1$.)

1.4 Möbius Inversion

We discuss in this section the Möbius matrices from a combinatorial view point. Let $(S, \le)$ be a finite partially ordered set (= poset). The Möbius-function of this poset is the function $\mu\colon S \times S \to \mathbb{Z}$ with the properties

$$\mu(x, x) = 1, \quad \textstyle\sum_{y, x \le y \le z} \mu(x, y) = 0 \quad \text{for } x < z, \quad \mu(x, y) = 0 \quad \text{for } x \not\le y.$$

These properties allow for an inductive computation of μ. We use the Möbius-function for the Möbius-inversion: Let $f, g\colon S \to \mathbb{Z}$ be functions such that

$$g(x) = \textstyle\sum_{y, x \le y} f(y). \tag{1.6}$$

Then

$$f(x) = \textstyle\sum_{y, x \le y} \mu(x, y) g(y). \tag{1.7}$$

A more general combinatorial formalism uses the associative incidence algebra with unit $I(S, \leq)$ of a poset. It consists of all functions $f\colon S \times S \to \mathbb{Z}$ such that $f(x,y) = 0$ if $x \not\leq y$ with pointwise addition and multiplication

$$(f * g)(x,y) = \textstyle\sum_{z, x \leq z \leq y} f(x,z)g(z,y).$$

(One can, more generally, define a similar algebra for functions into a commutative ring R.) The unit element of this algebra is the Kronecker-delta

$$\delta(x,y) = 1, \text{ for } x = y, \quad \delta(x,y) = 0, \text{ otherwise.}$$

If we define the function ζ by $\zeta(x,y) = 1$ for $x \leq y$, then the Möbius-function is the inverse of ζ in this algebra $\mu = \zeta^{-1}$. The group of functions $\alpha\colon S \to \mathbb{Z}$ becomes a left module over the incidence algebra via $(f * \alpha)(x) = \sum_y f(x,y)\alpha(y)$. We can now write ((1.6)) and ((1.7)) in the form $g = \zeta * f$, $f = \zeta^{-1} * g = \mu * g$.

Let G be a finite group. We apply this to the poset $(\mathrm{Sub}(G), \leq)$ and write $\mu(1,H) = \mu(H)$, with the trivial group 1. Conjugation of subgroups yields an action of G on this poset by poset automorphisms. Let $I^{Con}(G)$ denote the subalgebra of $I(\mathrm{Sub}(G), \leq)$ of invariant functions $f(K,L) = f(gKg^{-1}, gLg^{-1})$. We also have the poset $(\mathrm{Con}(G), \leq)$ of conjugacy classes. For $f \in I^{Con}(G)$ we define $f_*(K,L) = \sum\{f(A,L) \mid A \in (K,L)_*\}$ and $f^*(K,L) = \sum\{f(K,B) \mid B \in (K,L)^*\}$; see ((1.1)) and ((1.2)) for the notation. One verifies that $f_*(K,L)$ and $f^*(K,L)$ only depend on the conjugacy classes of K and L. Moreover:

(1.4.1) Proposition. *The assignments*

$$c_*\colon I^{Con}(G) \to I(\mathrm{Con}(G)),\ f \mapsto f_*, \quad c^*\colon I^{Con}(G) \to I(\mathrm{Con}(G)),\ f \mapsto f^*$$

are unital algebra homomorphisms. □

With these notations $c_*(\zeta) = \zeta_*$, $c^*(\zeta) = \zeta^*$ In particular, since $\zeta \in I^{Con}$, we have in $I(\mathrm{Con}(G))$ the inverses μ^* of ζ^* and μ_* of ζ_*. Recall that $\zeta^*(K,L) = |(K,L)^*|$ and $\zeta_*(K,L) = |(K,L)_*|$.

Let S be a finite G-set. Then we have $S^H = \coprod_{H \leq K} S_K$ and hence

$$|S_H| = \textstyle\sum_{K, H \leq K} \mu(H,K)|S^K|.$$

Since $S_{(H)} \cong G/H \times_{WH} S_H$, we see that the number $m_H(S)$ of orbits of type H in S is given by

$$m_H(S) = \tfrac{1}{|WH|} \textstyle\sum_{K, H \leq K} \mu(H,K)|S^K|;$$

note that $|S_{(H)}/G| = |S_H/WH|$, and WH acts freely on S_H. We rewrite this in terms of conjugacy classes:

$$m_H(S) = \tfrac{1}{|WH|} \textstyle\sum_{(K),(H) \leq (K)} \mu^*(H,K)|S^K|.$$

1.5 The Möbius Function

In this section we investigate the Möbius-function by combinatorial methods.

(1.5.1) Lemma. *Let* $1 \neq N \lhd G$ *and* $N \leq K \leq G$. *Then* $\sum_{X, XN=K} \mu(X) = 0$.

Proof. By definition of μ, this holds for $K = N$. We assume inductively, that the assertion holds for all proper subgroups Y of K which do not contain N. The computation

$$0 = \sum_{X \leq K} \mu(K) = \sum_{X, XN=K} \mu(X) + \sum_{N \leq Y < K} \left(\sum_{XN=Y} \mu(X) \right) = \sum_{X, XN=K} \mu(X)$$

yields the claim. □

(1.5.2) Proposition. *Let* $N \lhd G$ *and let* $Co(G,N) = \{K \leq G \mid KN = G, K \cap N = 1\}$ *be the set of complements of* N *in* G. *Then*

$$\mu(G) = \mu(G/N) \cdot \sum_{K \in Co(G,N)} \mu(K,G).$$

Proof. (An empty sum yields zero.) The assertion is trivial in the case that $N = 1$; hence assume $N > 1$. By (1.5.1),

$$\mu(G) = - \sum_{X<G, XN=G} \mu(X).$$

We use induction over the order of G. This yields for the summand $\mu(X)$

$$\mu(X) = \mu(X/X \cap N) \cdot \sum_{K \in Co(X, X \cap N)} \mu(K,X).$$

Since $XN = G$ we have $G/N \cong X/X \cap N$. Therefore $\mu(G)$ equals

$$-\mu(G/N) \cdot \textstyle\sum_{X<G, XN=G} \left(\displaystyle\sum_{K \in Co(X, X \cap N)} \mu(K,X) \right).$$

One verifies that the following conditions (1) and (2) on X, K are equivalent:

(1) $X < G$, $XN = G$, $K \in Co(X, X \cap N)$

(2) $K \leq X < G$, $K \in Co(G,N)$.

For (1) says $K \leq X < G, K \cap N = 1; XN = G, K \cdot (X \cap N) = X$, and (2) says $K \leq X < G, K \cap N = 1; K \cdot N = G$. In order to prove (1) ⇒ (2) we multiply the last equation in (1) with N. In order to prove (2) ⇒ (1) we use the modular property of the subgroup lattice which says in general terms: For $A, B, C \leq G$ and $A \leq C$ the equality $AB \cap C = A(B \cap C)$ holds. By definition of μ we know

$$\sum_{X, K \leq X < G} \mu(K,X) = -\mu(K,G).$$

Now we put everything together and obtain the claim. □

(1.5.3) Proposition. *Let G be a group with $\mu(G) \neq 0$. Let $N \leq M \lhd G$ and $N \lhd G$. Then M/N has a complement in G/N.*

Proof. By the previous note, $\mu(G/N) \neq 0$. Therefore it suffices to treat the case $N = 1$. But then, again by (1.5.2), $Co(G, M) \neq \emptyset$. □

The ***Frattini-subgroup*** $\Phi(G)$ of G is the intersection of its maximal subgroups. The first assertion of the next note follows immediately from the definition of $\Phi(G)$. For the second one we use the fact that a maximal subgroup of a p-group is a normal subgroup of index p. See [**?**, III.3.2 and III.3.14].

(1.5.4) Proposition. (1) *Let $N \lhd G$. Then there exists $H < G$ with $G = NH$ if and only if N is not contained in $\Phi(G)$.*
(2) *Let G be a p-group. Then $G/\Phi(G)$ is elementary abelian, and $\Phi(G)$ is the smallest normal subgroup N such that G/N is elementary abelian.* □

(1.5.5) Corollary. (1.5.3) *and* (1.5.4) *imply:*

(1) *Let $\mu(G) \neq 0$. Then $\Phi(G) = 1$.*
(2) *If G is a p-group and $\mu(G) \neq 0$, then G is elementary abelian.*

Proof. (1) Suppose $\mu(G) \neq 0$. Then we know from (1.5.3) that $\Phi(G)$ has a complement in G, and this is impossible, by (1.5.4)(1), if $\Phi(G) \neq 1$.

(2) If G is not elementary abelian, then $\Phi(G) \neq 1$, by (1.5.4)(2), and therefore $\mu(G) = 0$. □

We now reprove (1.3.5).

(1.5.6) Proposition. *Let P be a p-group. Then $\mu^*(1, H) \neq 0$ if and only if $H \leq P$ is elementary abelian. If H is elementary abelian of order $|H| = p^d$, then $\mu^*(1, H) = (-1)^d p^{d(d-1)/2} |P/NH|$.*

Proof. We have just seen a proof of the first assertion. It remains to determine $\mu(G)$ for elementary abelian G. We induct over $|G|$. Suppose $A \leq G$, $|A| = p$, and $|G| = p^d$. Among the $(p^d - 1)/(p - 1) = b$ maximal subgroups exactly $(p^{d-1} - 1)/(p - 1) = a$ contain the subgroup A. Therefore A has $b - a = p^{d-1}$ complements. By (1.5.2), $\mu(G) = -p^{d-1}\mu(G/A)$, since $\mu(K, G) = -1$ for a maximal subgroup K. □

Problems

1. The function $H \mapsto \mu(H, A_5)$ is displayed in the next table.

1	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	D_2	D_3	D_5	A_4	A_5
-60	4	2	0	0	-1	-1	-1	1

1.6 One-dimensional Representations

We study in some detail the simplest type of representations, namely one-dimensional representations of finite groups G over the complex numbers; these are just the homomorphisms $G \to \mathbb{C}^*$. These representations do not need much theory, and they will be used at various occasions, e.g., as input for the construction of more complicated representations (later called induced representations). The set $X(G) = G^*$ of these homomorphisms becomes an abelian group with product $(\alpha \cdot \beta)(g) = \alpha(g)\beta(g)$. A homomorphism $\varphi\colon A \to B$ induces a homomorphism $X(\varphi)\colon X(B) \to X(A)$, $\beta \mapsto \beta \circ \varphi$. In this manner X yields a contravariant functor from finite groups to finite abelian groups. The group $X(G)$ will be called the ***character group*** of G, and $\alpha \in X(G)$ is a (linear) ***character*** of G.

Since $\mathbb{C}^*$ is abelian, a homomorphism $\alpha\colon G \to \mathbb{C}^*$ maps the ***commutator subgroup*** $[G,G] = G'$, generated by the commutators $uvu^{-1}v^{-1}$, to 1 and induces $\overline{\alpha}\colon G/[G,G] \to \mathbb{C}^*$. The factor group $G/[G,G]$ is abelian and it is called the ***abelianized quotient*** G^{ab} of G.

For the cyclic group $C_m = \langle c \mid c^m = 1 \rangle$ the character group $X(C_m)$ is the cyclic group of order m generated by $\rho\colon c \mapsto \exp(2\pi i/m)$. Let G and H be groups. Let $\alpha\colon G \to \mathbb{C}^*$ and $\beta\colon H \to \mathbb{C}^*$ be homomorphisms. Then $\alpha \odot \beta\colon G \times H \to \mathbb{C}^*$, $(g,h) \mapsto \alpha(g)\beta(h)$ is again a homomorphism, and

$$\odot\colon G^* \times H^* \longrightarrow (G \times H)^*, \quad (\alpha, \beta) \longmapsto \alpha \odot \beta$$

is a homomorphism between character groups. One verifies that $\odot$ is an isomorphism[1].

For a finite abelian group A the group A^* is isomorphic to A. This follows from the previous remarks and the structure theorem about finite abelian groups which says that each such group is isomorphic to a product of cyclic groups.

Let $H \lhd G$ be a normal subgroup of G. Then the group G acts as a group of automorphisms on $X(H)$ by $(g \cdot \gamma)(h) = \gamma(ghg^{-1})$. If γ is the restriction of a homomorphism $\alpha \in X(G)$, then $g \cdot \gamma = \gamma$. Therefore the restriction homomorphism $X(G) \to X(H)$ has an image in the fixed point group $X(H)^G$; its elements are called G-invariant. Note that the G-action on $X(H)$ factors over G/H.

Let G be the semi-direct product of the normal subgroup A and a group P, i.e. $G = AP$ and $A \cap P = 1$. Let

$$\Gamma\colon X(G) \to X(A)^P \times X(P)$$

be the product of the restriction homomorphisms.

---

[1]Recall the categorical notion: sum in the category of abelian groups.

(1.6.1) Proposition. Γ *is an isomorphism.*

Proof. Since $\gamma \in X(G)$ is determined by the restrictions to A and P, the map Γ is injective. Given $\alpha \in X(A)^P$, i.e. $\alpha(a) = \alpha(xax^{-1})$ for $a \in A, x \in P$, and $\beta \in X(P)$. Define a map $\gamma\colon G \to \mathbb{C}^*$ by $\gamma(ax) = \alpha(a)\beta(x)$. We verify that γ is a homomorphism

$$\begin{aligned}\gamma(axa_1x_1) &= \gamma(axa_1x^{-1}xx_1)\\ &= \alpha(axa_1x^{-1})\beta(xx_1)\\ &= \alpha(a)\alpha(xa_1x^{-1})\beta(x)\beta(x_1)\\ &= \alpha(a)\beta(x)\alpha(a_1)\beta(x_1)\\ &= \gamma(ax)\gamma(a_1x_1).\end{aligned}$$

By construction, $\Gamma(\gamma) = (\alpha, \beta)$. □

A homomorphism $\alpha \in X(A)$ is P-invariant if and only if it vanishes on the normal subgroup A_P generated by the elements $axa^{-1}x^{-1}$ for $a \in A$ and $x \in P$. Thus the quotient map $\pi\colon A \to A/A_P$ induces an isomorphism $X(A/A_P) \to X(A)^P$.

(1.6.2) Proposition. *Let the group P act on the abelian group A by automorphisms $(x, a) \mapsto x \diamond a$. Suppose $(|A|, |P|) = 1$. Let $A_P \le A$ denote the subgroup generated by the elements $a \cdot (x \diamond a)^{-1}$. Then the inclusion $\iota\colon A^P \mapsto A/A_P$ is an isomorphism.*

Proof. For $a \in A$ set $\mu(a) = \prod_{x\in P}(x \diamond a)$. Then $\mu(a) \in A^P$, and for $a \in A^P$ we have $\mu(a) = a^{|P|}$. Since $|P|$ is prime to the order of A, the map $a \mapsto a^{|P|}$ is an automorphism of A and A^P. The group A_P is contained in the kernel of μ, since, by construction, $\mu(y \diamond a) = \mu(a)$ for $a \in A$ and $x \in P$. Hence we obtain an induced map $\nu\colon A/A_P \to A^P$, and $\nu \circ \iota$ is an isomorphism. On the other hand $\iota\nu(a) = a^{|P|}\prod((x \diamond a)a^{-1})$, and this shows that $\iota \circ \nu$ is an isomorphism too. □

As a consequence of (1.6.1) and (1.6.2) we obtain the next result which will later be used in the proof of the Brauer induction theorem (4.6.5).

(1.6.3) Proposition. *Let $G = AP$ be the semi-direct product of the abelian subgroup A by P. Suppose $(|A|, |P|) = 1$. Then the restriction $X(G) \to X(A^P \times P)$ is an isomorphism.* □

We continue the study of one-dimensional representations and demonstrate their use in group theory. Let $H \le G$ and $\alpha \in X(H)$. We associate to α an element $m_H^G\alpha \in X(G)$. For this purpose we choose a representative system

$g_1, \dots, g_r$ of G/H. For each $g \in G$ we have $gg_i = g_{\sigma(i)} h_i$ with a permutation $\sigma \in S_r$ and certain $h_i \in H$. We set

$$(m_H^G \alpha)(g) = \prod_{i=1}^{r} \alpha(g_{\sigma(i)}^{-1} g g_i) = \prod_{i=1}^{r} \alpha(h_i).$$

One verifies that $m_H^G \alpha$ is a well-defined homomorphism and that, moreover, $m_H^G \colon X(H) \to X(G)$ is a homomorphism. We call it ***multiplicative induction***.

We use this construction to deal with the question: Given $\alpha \in X(H)$, when does there exist an extension $\beta \in X(G)$ such that $\beta | H = \alpha$? Suppose it exists. Then for $h \in H$, $u \in G$ and $g = uhu^{-1}$ we have $\beta(g) = \beta(h)$. Thus a necessary condition for the existence of β is that α is trivial on the subgroup H_0 generated by $\{xy^{-1} \mid x \sim_G y, x, y \in H\}$.

(1.6.4) Proposition. *Suppose H and G/H have coprime order. Then an extension β exists if and only if α vanishes on H_0.*

Proof. Let α have the stated property. We compute $m_H^G \alpha(h)$ for $h \in H$. For this purpose we make a special choice of the coset representatives: The cyclic group $\langle h \rangle$ acts on G/H; let $gH, hgH, \dots, h^{t-1}gH$ be an orbit, and suppose $h^t g = g\tilde{h}$. Then this orbit contributes $\alpha(\tilde{h}) = \alpha(g^{-1}h^t g) = \alpha(h)^t$ to the product in the definition of $m_H^G \alpha$. Altogether we obtain $\mathrm{res}_H^G \, m_H^G \alpha = \alpha^{|G/H|}$. Hence if H and G/H have coprime order, then $\mathrm{res}_H^G \, m_H^G$ is an automorphism because $|X(H)|$ is coprime to $|G/H|$. Therefore there exists an extension. □

(1.6.5) Proposition. *Let H be a Sylow p-subgroup of G. Then $H_0 = H \cap G'$.*

Proof. If $y = g^{-1}xg$, then $xy^{-1} = xg^{-1}x^{-1}g$, so that $P' \le P_0 \le P \cap G'$. It remains to show $P \cap G' \le P_0$. Given $x \in P \setminus P_0$, there exists $\lambda \in X(P)$ such that $\lambda(x) \neq 1$ with trivial $\lambda | P_0$. By the previous proposition, λ has an extension $\theta \colon G \to \mathbb{C}^*$. Since $\theta(x) \neq 1$, we see that $x \notin P \cap G'$. □

The previous considerations lead to a simple proof of the so-called ***normal complement theorem***.

(1.6.6) Proposition. *Let $G(p)$ be an abelian Sylow p-group of G and assume that $NG(p) = G(p)$. Then there exists a normal subgroup $H \lhd G$ such that $N \cap G(p) = 1$.*

Proof. The quotient G/G' has Sylow group $G(p)$ if $G' \cap P = P_0 = 1$. This means: Suppose $x, y \in P$ are conjugate in G, then $x = y$. This is a consequence of the next lemma. Since G/G' is abelian, there exists a complement of $G(p)$, and the pre-image in G is the required complement. □

(1.6.7) Lemma. *Let G have abelian Sylow p-group P. Suppose $x, y \in P$ are conjugate in G. Then they are conjugate in $N_G P$.*

Proof. Let $y = gxg^{-1}$. Since P is abelian, P is a subgroup of the centralizer $C_G(y)$ of y in G, moreover $g(C_G(x))g^{-1} = C_G(y)$. Hence gPg^{-1} and P are Sylow groups of $C_G(y)$. Therefore there exists $n \in C_G(y)$ such that $ngPg^{-1}n^{-1} = P$. Hence $ng \in NG(P)$ and $y = ny^{-1} = ngxg^{-1}n^{-1}$. □

Problems

1. Let B be a subgroup of the finite abelian group A. Show that for each $a \in A \smallsetminus B$ there exists $\alpha \in X(A)$ with $\alpha(a) \neq 1$.
2. The isomorphism between G and G^* is not natural, but there exists a canonical and natural isomorphism $G \to X(X(G))$, G abelian. (This is analogous to the double dual of finite dimensional vector spaces.)
3. Let $1 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 1$ be an exact sequence of finite abelian groups. Then the functor X transforms it into an exact sequence. Exactness at $X(B)$ and $X(C)$ is formal; for the exactness at $X(A)$ one can use the knowledge of the order of this group.

1.7 Representations as Modules

The vector space KG has more structure than just carrying the left and right regular representation.

There is a bilinear map $KG \times KG \to KG$ which extends the group multiplication $(g, h) \mapsto gh$ of the basis elements. This bilinear map defines on KG the structure of an associative algebra with unit. This algebra is called the ***group algebra*** KG of G over K. The multiplication in the group algebra is therefore defined by the formula

$$(\textstyle\sum_{g\in G} \lambda(g)g) \cdot (\sum_{h\in G} \mu(h)h) = \sum_{g,h} \lambda(g)\mu(h)gh = \sum_{u\in G} \gamma(u)u$$

with $\gamma(u) = \sum_{g\in G} \lambda(g)\mu(g^{-1}u)$. Another model for the group algebra is the vector space $C(G, K)$ of functions $G \to K$ with ***convolution product***

$$(\alpha * \beta)(u) = \textstyle\sum_{g\in G} \alpha(g^{-1})\beta(u^{-1}g).$$

The assignment $C(G, K) \to KG, \varphi \mapsto \sum_g \varphi(g^{-1})g$ is an isomorphism of algebras. Under this isomorphism the natural left-right action on $C(G, K)$, given by

$$(g \cdot \varphi \cdot h)(x) = \varphi(hxg),$$

corresponds to the left-right action on KG.

(1.7.1) Example. The group algebra of the cyclic group $C_n = \langle\, x \mid x^n = 1 \,\rangle$ is the quotient $K[x]/(x^n - 1)$ of the polynomial algebra $K[x]$ by the principal ideal $(x^n - 1)$. ◇

We now come to the third form of a representation, that of a module over the group algebra. Let V be a KG-representation. The bilinear map

$$KG \times V \to V, \quad (\sum_g \lambda(g)g, v) \mapsto \sum_g \lambda(g)(g \cdot v)$$

is the structure of a unital KG-module on the vector space V. The element $\sum_g \lambda(g)g \in KG$ acts on V as the linear combination $\sum_g \lambda(g)g$. A morphism $V \to W$ of representations becomes a KG-linear map. Conversely, given a KG-module M we obtain a representation on M by defining l_g as the scalar multiplication by $g \in KG$ in the module. In this manner, the category KG-Rep of finite-dimensional KG-representations becomes the category KG-Mod of left KG-modules which are finite-dimensional as vector spaces. Direct sums correspond in these categories. A module M over an algebra A is ***irreducible***, if it has no submodules different from 0 and M.

The view point of modules allows for an algebraic construction of representations. Consider KG as a left module over itself. Then a left ideal is a representation. A non-zero left ideal yields an irreducible module, if it is a minimal left ideal with respect to inclusion. Let M be an irreducible KG-module and $0 \neq x \in M$. Then $KG \to M, \lambda \mapsto \lambda x$ is KG-linear; its kernel I is a left ideal and the induced map $KG/I \to M$ an isomorphism, since M is irreducible; the ideal I is then a maximal ideal.

(1.7.2) Example. The maximal ideals in the group algebra $KC_n = K[x]/(x^n - 1)$ correspond to principal ideals $(q) \subset K[x]$ where q is an irreducible factor of $x^n - 1$. If K is a splitting field for $x^n - 1$, then the irreducible factors are linear, and irreducible representations one-dimensional. Over $\mathbb{Q}$, the polynomial is the product $\prod_{d|n} \Phi_d(x)$ of the irreducible cyclotomic polynomials Φ_d. The complex roots of Φ_d are the primitive d-th roots of unity. As an example

$$x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1).$$

The representation on $\mathbb{Q}[x]/(x^2 - x + 1)$ is given in the basis $1, x$ by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus we know that this matrix has order 6; this can, of course, be checked by a calculation. One the other hand, it is a nontrivial task to find matrices in $SL_2(\mathbb{Z})$ of order 6. ◇

1.8 Linear Algebra of Representations

Standard constructions of linear algebra may be used to obtain new representations from old ones. We begin with direct sums.

Let $V_1, \dots V_r$ be vector spaces over K. Their (external) direct sum $V_1 \oplus \cdots \oplus V_r$ consists of all r-tuples $(v_1, \dots, v_r)$, $v_j \in V_j$ with component-wise addition and scalar multiplication. If the V_j are subspaces of a vector space V, we say, V is the (internal) direct sum of these subspaces, if each $v \in V$ has a unique presentation of the form $v = \sum_{j=1}^r v_j$ with $v_j \in V_j$. We also use the notation $V = \oplus_{j=1}^r V_j$, because V is canonically isomorphic to the external direct sum of the V_j. A subspace U of W is a direct summand, if there exists a complementary subspace V, i.e., a subspace V such that $U \oplus V = W$.

Let $(V_j \mid j \in J)$ be a family of subspaces of V. The sum $\sum_{j \in J} V_j$ is the subspace of V generated by the V_j. It is the smallest subspace containing the V_j and consists of the elements which are sums of elements in the various V_j. We use the following fact from linear algebra.

(1.8.1) Proposition. *Let $V_1, \dots, V_n$ be subspaces of V. The following are equivalent:*

(1) *V is the internal direct sum of the V_j.*

(2) *V is the sum of the V_j, and $V_j \cap \sum_{i \neq j} V_i = \{0\}$ for all j.* □

We now apply these concepts to representations. We use two simple observations. If $(V_j \mid j \in J)$ are sub-representations of V, then their sum is again a sub-representation. The direct sum $U \oplus V$ of representations becomes a representation with respect to the component-wise group action $g \cdot (u, v) = (g \cdot u, g \cdot v)$. Similarly for an arbitrary number of summands. This defines the ***direct sum*** of representations. If $g \mapsto A(g)$ and $g \mapsto B(g)$ are matrix representations associated to U and V, then the block matrices

$$\begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}$$

yield a matrix representation for $U \oplus V$. A representation is called ***indecomposable***, if it is not the direct sum of non-zero sub-representations. An irreducible representation is clearly indecomposable, but the converse does not hold in general.

(1.8.2) Example. In (1.1.4) we defined two sub-representations T_n, D of the permutation representation of S_n on K^n. Given $(x_1, \dots, x_n) \in K^n$ write $x = n^{-1} \sum_j x_j$. Then $(x_1 - x, \dots, x_n - x) \in T_n$ and $(x, \dots, x) \in D$. Hence $T_n + D = K^n$. The intersection $T_n \cap D$ consists of the $(y, \dots, y)$ with $ny = 0$. This implies $y = 0$. Hence $T_n \oplus D = K^n$. But note: This argument requires that n^{-1} makes sense in K, i.e., the characteristic of K does not divide n.

If $n = 2$ and $K = \mathbb{F}_2$ is the field with two elements, then $V = W$! The regular representation is not irreducible, because it has a one-dimensional fixed point set. If this fixed point set had a complement it would be a one-dimensional representation, hence a trivial representation. Therefore the regular representation is indecomposable. ◇

Here is another result from linear algebra.

(1.8.3) Proposition. *A sub-representation W of V is a direct factor if and only if there exists a projection morphism $q\colon V \to V$ with image W. A projection is a morphism q such that $q \circ q = q$. If q is a projection, then V is the direct sum of the image and the kernel of q.* □

Let V and W be representations of G. The ***tensor product*** representation $V \otimes_K W$ has the action $g(v \otimes w) = gv \otimes gw$. If $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W, then the $v_i \otimes w_k$ form a basis of $V \otimes W$. The map $V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$ is bilinear. If g acts on V and W via matrices (r_{ij}) and (s_{kl}), then g acts on $V \otimes W$ via the matrix $(r_{ij}s_{kl})$ whose entry in the (i, k)-th row and (j, l)-th column is $r_{ij}s_{kl}$. More explicitely, if $gv_j = \sum_i r_{ij}v_i$ and $gw_l = \sum_k s_{kl}w_k$, then

$$g(v_j \otimes w_l) = \textstyle\sum_{i,k} r_{ij}s_{kl}v_i \otimes w_k.$$

If V is one-dimensional and given by a homomorphism $\alpha\colon G \to K^*$, then we simply multiply the matrix (s_{kl}) with $\alpha(g)$ in order to obtain the tensor product.

Let V and W be G-representations. We have a G-action on the vector space $\mathrm{Hom}(V, W)$ of K-linear maps, given by $(g \cdot \varphi)(v) = g\varphi(g^{-1}v)$. The fixed point set is $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. When $W = K$ is the trivial representation we obtain the ***dual representation*** $V^* = \mathrm{Hom}(V, K)$ of V. If $g \mapsto A(g)$ is the matrix representation of V with respect to a basis, then $g \mapsto {}^tA(g)^{-1}$ (inverse of the transpose) is the matrix representation of V^* with respect to the dual basis.

(1.8.4) Note. *There is a canonical isomorphism*

$$V^* \otimes W \xrightarrow{\cong} \mathrm{Hom}(V, W), \quad \varphi \otimes w \mapsto (u \mapsto \varphi(u)w).$$

One verifies that it is a morphism of G-representations. □

In some of the constructions one can also use representations for different groups. Let V be a G-representation and W an H-representation. Then $V \otimes W$ becomes a $G \times H$-representation via $(g, h)(v \otimes w) = gv \otimes hw$. Similarly, we have a $G \times H$-action on $\mathrm{Hom}(V, W)$ defined as $((g, h) \cdot \psi)(v) = h\psi(g^{-1}v)$. With these actions, (1.8.4) is an isomorphism of $G \times H$-representations.

(1.8.5) Example. Let S and T be finite G-sets. There are canonical isomorphisms

$$K(S \amalg T) \cong K(S) \oplus K(T), \quad K(S \times T) \cong K(S) \otimes K(T), \quad K(S)^* \cong K(S).$$

They are induced by a G-equivariant bijection of the canonical bases. We combine with (1.8.4) and obtain

$$\mathrm{Hom}(K(S), K(T)) \cong K(S)^* \otimes K(T) \cong K(S) \otimes K(T) \cong K(S \times T).$$

Together with (1.2.2) we get

$$\dim_K \mathrm{Hom}_G(K(S), K(T)) = |(S \times T)/G|.$$

The representation (1.1.4) of S_n on K^n by permutation of coordinates (made into a left representation by inversion) is isomorphic to $K(S_n/S_{n-1})$ where S_{n-1} is the subgroup of S_n which fixes $1 \in \{1, \ldots, n\}$. The action of S_{n-1} on S_n/S_{n-1} has two orbits, of length 1 and $n-1$. Hence $\mathrm{Hom}_{S_n}(K^n, K^n)$ is two-dimensional. ◇

1.9 Semi-simple Representations

The topic of this section is the decomposition of a representation into a direct sum of sub-representations.

We begin with a simple and typical example. Let $\alpha\colon G \to K^*$ be a homomorphism. Consider $x_\alpha = \sum_{g \in G} \alpha(g^{-1})g \in KG$. The computation

$$h \cdot x_\alpha = \textstyle\sum_g \alpha(g^{-1})hg = \sum_g \alpha(h)\alpha(g^{-1}h^{-1})hg = \alpha(h)x_\alpha$$

shows that x_α spans a one-dimensional sub-representation $V(\alpha)$ of the regular representation. Let $K = \mathbb{C}$ and $G = C_n = \langle a \mid a^n = 1 \rangle$ the cyclic group. There are n different homomorphisms $\alpha(j)\colon C_n \to \mathbb{C}^*, 1 \le j \le n$. The vectors $x_{\alpha(j)}$ are different eigenvectors of l_a. Therefore we have a decomposition $\mathbb{C}C_n = \oplus_j V(\alpha(j))$ into one-dimensional representations. A similar decomposition exists for finite abelian groups G, since we still have $|G|$ homomorphisms $G \to \mathbb{C}^*$. Our aim is to find analogous decompositions for general finite groups.

(1.9.1) Theorem. *Let V be the sum of irreducible representations $(U_j \mid j \in J)$ and let U be a sub-representation. Then there exist a finite subset $E \subset J$ such that V is the direct sum of U and the $U_j, j \in E$.*

Proof. If $W \neq V$ is any sub-representation, then there exists $k \in J$ such that $V_k \not\subset W$, since V is the sum of the V_j. Then $V_k \cap W = 0$, and $W + V_k = W \oplus V_k$. If now $E \subset J$ is a maximal subset such that the sum W of U and the $V_j, j \in E$ is direct, then necessarily $W = V$. □

(1.9.2) Theorem. *The following assertions about a representation M are equivalent:*

(1) *M is a direct sum of irreducible sub-representations.*
(2) *M is a sum of irreducible sub-representations.*
(3) *Each sub-representation is a direct summand.*

Proof. (1) $\Rightarrow$ (2) as special case; and (2) $\Rightarrow$ (3) is a special case of (1.9.1).

(3) $\Rightarrow$ (1). Let $\{M_1, \dots, M_n\}$ be a set of irreducible sub-representations such that their sum N is the direct sum of the M_j. If $N \neq M$ then, by hypothesis, there exists a sub-representation L such that $M = N \oplus L$. Each sub-representation contains an irreducible one. If $M_{n+1} \subset L$ is irreducible, then the sum of the $\{M_1, \dots, M_{n+1}\}$ is direct. □

A representation is called ***semi-simple*** or ***completely reducible*** if it has one of the properties (1)-(3) in (1.9.2).

(1.9.3) Proposition. *Sub-representations and quotient representations of semi-simple representations are semi-simple.*

Proof. Let M be semi-simple and $F \subset N \subset M$ sub-representations. A projection $M \to M$ with image F restricts to a projection $N \to N$ with image F. Hence F is a direct summand in N.

Suppose $N \oplus P = M$; then the quotient $M/N \cong P$ is semi-simple. □

(1.9.4) Proposition. *Let V be the sum of irreducible sub-representations $(V_j \mid J)$. Then each irreducible sub-representation W is isomorphic to some V_j.*

Proof. There exists a surjective homomorphism $\beta\colon V \to W$, by (1.8.3) and (1.9.2). If W were not isomorphic to some V_j, then the restriction of β to each V_j would be zero, by Schur's lemma, hence β would be the zero morphism. □

We write

$$\langle U, V \rangle = \dim_K \mathrm{Hom}_G(U, V)$$

for G-representations U and V. This integer depends only on the isomorphism classes of U and V. Note the additivity $\langle U_1 \oplus U_2, V \rangle = \langle U_1, V \rangle + \langle U_2, V \rangle$, and similarly for the second argument.

(1.9.5) Proposition. *Suppose $V = V_1 \oplus \cdots \oplus V_r$ is a direct sum of irreducible representations V_j. Let W be any irreducible representation and denote by $n(W,V)$ the number of V_j which are isomorphic to W. Then*

$$\langle W, W \rangle n(W, V) = \langle W, V \rangle = \langle V, W \rangle.$$

Therefore $n(W, V)$ is independent of the decomposition of V into irreducibles.

Proof. For a direct sum as above, we have a canonical isomorphism

$$\mathrm{Hom}_G(W,V) \cong \textstyle\prod_{j=1}^r \mathrm{Hom}_G(W,V_j).$$

This expresses the fact that a morphism $W \to V$ is nothing else but an r-tuple of morphisms $W \to V_j$. The assertion is now a direct consequence of Schur's lemma. For the second assertion we use the canonical isomorphism $\mathrm{Hom}_G(V,W) \cong \prod_j \mathrm{Hom}_G(V_j,W)$. (In conceptual terms: We are using the fact that $\oplus_j V_j$ is the sum and the product of the V_j in the category of representations.) □

We call the integer $n(W,V)$ in (1.9.5) the ***multiplicity*** of the irreducible representation W in the semi-simple representation V. We say W ***occurs*** in V or is ***contained*** in V if $n(W,V) \neq 0$. In fact, if $n(W,V) \neq 0$, then V has a sub-representation which is isomorphic to W: take a non-zero morphism $W \to V$ and apply Schur's lemma. The irreducible representation W appears in V if and only if $\mathrm{Hom}_G(W,V)$ or $\mathrm{Hom}_G(V,W)$ is non-zero.

Let W be irreducible and denote by $V(W)$ the sum of the irreducible sub-representations of V which are isomorphic to W. We call $V(W)$ the W-***isotypical*** part of V, if $V(W) \neq 0$, and the decomposition in (1.9.6) is the ***isotypical decomposition*** of V. Let $I = \mathrm{Irr}(G;K)$ denote a complete set of pairwise non-isomorphic irreducible representations of G over K.

(1.9.6) Theorem. *A semi-simple representation V is the direct sum of its isotypical parts.*

Proof. Since V is semi-simple it is the direct sum of irreducible sub-representations and therefore the sum of its isotypical parts. Let $A \in I$ and let Z be the sum of the $V(B), B \in I, B \neq A$. We refer to (1.8.1) and have to show: $V(A) \cap Z = 0$. Suppose this is not the case. Then the intersection would contain an irreducible sub-representation, and by (1.9.4) it would be isomorphic to A and to some $B \neq A$. Contradiction. □

1.10 The Regular Representation

We now consider KG as left and right regular representation. For each representation U the vector space $\mathrm{Hom}_G(KG,U)$ becomes a left G-representation via $(g \cdot \varphi)(x) = \varphi(x \cdot g)$.

(1.10.1) Lemma. *The evaluation* $\mathrm{Hom}_G(KG,U) \to U$, $\varphi \mapsto \varphi(e)$ *is an isomorphism of representations.*

Proof. We use the fact that KG is a free KG-module with basis e. It is verified from the definitions that the evaluation is a morphism. Clearly, a morphism $KG \to U$ is determined by its value at e, and this value can be any prescribed element of U. □

(1.10.2) Theorem. *Suppose the left regular representation is semi-simple. Then each irreducible representation U appears in KG with multiplicity $n_U = \langle U, U \rangle^{-1} \dim_K U$.*

Proof. Since $U \cong \mathrm{Hom}_G(KG, U)$ is non-zero, each irreducible representation U appears in KG, see the remarks after (1.9.5). Suppose $KG \cong \bigoplus_{W \in I} n_W W$ where $n_W W$ denotes the direct sum of n_W copies of W. Then

$$\dim U = \langle KG, U \rangle = \textstyle\sum_{W \in I} n_W \langle W, U \rangle = n_U \langle U, U \rangle,$$

the latter by Schur's lemma. □

(1.10.3) Proposition. *Suppose the left regular representation is semi-simple.*

(1) *The number of isomorphism classes of irreducible representations is finite.*
(2) $|G| = \sum_{V \in I} \langle V, V \rangle^{-1} (\dim V)^2$.
(3) *If K is algebraically closed, then $|G| = \sum_{V \in I} (\dim V)^2$.*

Proof. (1) is a corollary of (1.10.2).

(2) Let $KG \cong \bigoplus_{W \in I} n_W W$. We insert the values of n_W obtained in (1.10.2).

(3) If the field K is algebraically closed then, by Schur's lemma, $\langle V, V \rangle = 1$ for an irreducible representation V. □

Part (3) of (1.10.3) gives us a method to decide whether a given set of pairwise non-isomorphic irreducible representations is complete. If G is abelian then irreducible representations over $\mathbb{C}$ are 1-dimensional. By (1.10.3) we see that there are $|G|$ non-isomorphic such representations; we know this, of course, from a direct elementary argument.

(1.10.4) Proposition. *A finite group G is abelian if and only if the irreducible complex representations are one-dimensional.*

Proof. A one-dimensional complex representation is given by a homomorphism $G \to \mathbb{C}^*$. The regular representation is faithful. If the regular representation is a sum of one-dimensional representations, then G has an injective homomorphism into an abelian group. The reversed implication was proved in (1.1.3). □

There remains the question: When is KG semi-simple? Recall some elementary algebra. For $n \in \mathbb{N}$ and $x \in K$, an expression nx stands for an n-fold sum $x + \cdots + x$. A relation $nx = 1$ exists in K if and only if either K has characteristic zero or the characteristic $p > 0$ of K does not divide n. In this case we say, n is invertible in K. We denote this inverse as usual by n^{-1}.

(1.10.5) Proposition. *If KG is semi-simple, then $|G|$ is invertible in K.*

Proof. Suppose KG is semi-simple. Then the fixed set $F = \{\lambda\Sigma \mid \lambda \in K, \Sigma = \sum_{g \in G} g\}$ is a sub-representation. Hence there exists a projection $p\colon KG \to F$. Since p is G-equivariant, it is determined by the value $p(e)$, say $p(e) = \mu\Sigma$. Since p is a projection we obtain $\Sigma = p(\Sigma) = \sum_{g \in G} p(e) = \sum_{g \in G} \mu\Sigma = |G|\mu\Sigma$. Thus we have shown: If KG is semi-simple, then $|G|$ is invertible in K. □

(1.10.6) Theorem (Maschke). *Suppose $|G|$ is invertible in K. Then G-representations are semi-simple.*

Proof. We show that each sub-representation W of a representation V is a direct summand (see (1.9.2)). There certainly exists a K-linear projection $p\colon V \to V$ with image W. We make it equivariant by an averaging process. Namely we define

$$q(v) = \tfrac{1}{|G|} \textstyle\sum_{g \in G} g^{-1} p(gv).$$

At this point we use the fact that $|G|^{-1}$ makes sense in K. By construction, q is K-linear as a linear combination of linear maps. For $h \in G$ we compute

$$q(hv) = \tfrac{1}{|G|} \textstyle\sum_{g \in G} g^{-1} p(ghv) = \tfrac{1}{|G|} h \sum_{g \in G} h^{-1} g^{-1} p(ghv) = hq(v),$$

and this verifies the equivariance. By hypothesis, $p(w) = w$ for $w \in W$, hence $p(gw) = gw$ and therefore $q(w) = w$. The values $q(v)$ are contained in W, hence $W = q(V)$ and $q^2 = q$. □

(1.10.7) Proposition. *Suppose V is semi-simple. Then $\langle V, V \rangle = 1$ implies that V is irreducible.*

Proof. Decompose into irreducibles $V = \sum n_W W$. Then $1 = \langle V, V \rangle = \sum n_W^2 \langle W, W \rangle$, by Schur's lemma. Hence one of the n_W is 1 and the others are 0. □

Assume that $|G|$ is invertible in K. In order that (1.10.3) holds, it is necessary to assume that $\langle V, V \rangle = 1$ for each $V \in I$. If $K \subset L$ is a field extension, then we can view a K-representation as an L-representation (just take the same matrices). However, an irreducible representation over K may become reducible over a larger field. This already happens for cyclic groups, as we have seen in the first section. If KG is semi-simple, then also LG. If the relation (1.10.3) holds for K-representations, then it also holds for L-representations.

The relation (1.10.3) is equivalent to $\langle V, V \rangle = 1$ for all $V \in I$. If this is the case, we call K a ***splitting field*** for G.

We now present the isotypical decomposition in a more canonical form. Let V be semi-simple. For each $U \in I$ we let $D(U)$ be its endomorphism algebra. Evaluation of endomorphisms makes U into a left $D(U)$-module. The vector space $\mathrm{Hom}_G(U, V)$ becomes a right $D(U)$-module via composition of endomorphisms. The evaluation $\mathrm{Hom}_G(U, V) \otimes U \to V, \varphi \otimes u \mapsto \varphi(u)$ induces a linear map $\iota_U \colon \mathrm{Hom}_G(U, V) \otimes_{D(U)} U \to V$.

(1.10.8) Theorem. *Let* $\iota\colon \bigoplus_{U \in I} \mathrm{Hom}_G(U, V) \otimes_{D(U)} U \to V$ *have components* ι_U. *Then* ι *is an isomorphism. The image of* ι_U *is the* U*-isotypical component of* V.

Proof. The maps ι constitute, in the variable V, a natural transformation on the category of semi-simple representations, and they are compatible with directs sums. Thus it suffices to consider irreducible V. In that case, by Schur's lemma, only the summand $\mathrm{Hom}_G(V, V) \otimes_{D(V)} V$ is non-zero, and evaluation is the canonical isomorphism $D(V) \otimes_{D(V)} V \cong V$. By construction, ι_U has an image in the U-isotypical part. □

Problems

1. Let $U^* = \mathrm{Hom}_K(U, K)$ be the dual vector space. This becomes a right G-representation via $(g \cdot \varphi)(u) = \varphi(gu)$. The vector space $\mathrm{Hom}_G(U, KG)$ becomes a right representation via $(\varphi \cdot g)(u) = \varphi(u) \cdot g$. Show: The linear map

$$U^* \to \mathrm{Hom}_G(U, KG), \quad \varphi \mapsto (u \mapsto \textstyle\sum_{g \in G} \varphi(g^{-1}u)g)$$

is an isomorphism of right representations. An inverse morphism assigns to $\alpha \in \mathrm{Hom}_G(U, KG)$ the linear form $U \to K$ which maps u to the coefficient of e in $\alpha(u)$.

Let KG be semi-simple. Then the isotypical decomposition (**??**) of KG assumes the form

$$\textstyle\bigoplus_{U \in I} U^* \otimes_{D(U)} U \to KG, \quad \varphi \otimes u \mapsto \sum_{g \in G} \varphi(g^{-1}u)g.$$

This is an isomorphism of left and right G-representations.

2. Use (1.10.3) in order to show that we found (in section 1) enough irreducible complex representations of the dihedral group D_{2n}. In the case that n is odd there are 2 one-dimensional and $(n-1)/2$ two-dimensional irreducibles. In the case that n is even there are 4 one-dimensional and $n/2 - 1$ two-dimensional irreducibles.

3. Use (1.10.7) in order to show that the representation of S_n on $T_n = \{(x_i) \in K^n \mid \sum_i x_i = 0\}$ by permutation of coordinates is irreducible (K characteristic zero).

4. Let V be a KG-representation. Let V_G denote the subrepresentation spanned by the vectors $v - gv, v \in V, g \in G$. Consider $\alpha\colon V^G \to V/V_G$ induced by the inclusion $V^G \subset V$. Show that α is an isomorphism if the characteristic of K does not divide $|G|$, and give an example where α is not bijective.

Chapter 2

Characters

2.1 Characters

We assume in this chapter that K has characteristic zero. It is then no essential restriction to assume moreover that $\mathbb{Q}$ is a subfield of K.

(2.1.1) Proposition. *Let U be a G-representation. Then the linear map*

$$p\colon U \to U, \quad u \mapsto |G|^{-1} \textstyle\sum_{g\in G} gu$$

is a G-equivariant projection onto the fixed point space U^G.

Proof. The map p is the identity on U^G, equivariant by construction, and the image is contained in U^G. □

Let V be a G-representation. We denote the trace of $l_g\colon V \to V$ by $\chi_V(g)$. The ***character*** of V is the function $\chi_V\colon G \to K$, $g \mapsto \chi_V(g)$. The character of an irreducible representation is an ***irreducible character***.

The trace of a projection operator is the dimension of its image. Therefore (2.1.1) yields the identity

$$\dim U^G = |G|^{-1} \textstyle\sum_{g\in G} \chi_U(g). \tag{2.1}$$

Recall from linear algebra: The trace of a matrix is the sum of the diagonal elements, and conjugate matrices have the same trace. If we express l_g in matrix form with respect to a basis, then the trace does not depend on the chosen basis. Since conjugate matrices have the same trace, isomorphic representations have the same character (1.1.1). Conjugation invariance also yields:

$$\chi_V(ghg^{-1}) = \chi_V(h), \qquad g,h \in G.$$

Thus characters are class functions. From the matrix form of representations we derive some properties of characters:

$$\chi_{V\oplus W} = \chi_V + \chi_W, \tag{2.2}$$
$$\chi_{V\otimes W} = \chi_V \chi_W, \tag{2.3}$$
$$\chi_{V^*}(g) = \chi_V(g^{-1}). \tag{2.4}$$

Let V and W be G-representations. In section 1.8 we introduced the representation $\mathrm{Hom}_K(V,W)$ with fixed point set $\mathrm{Hom}_{KG}(V,W)$.

(2.1.2) Proposition. *The character of* $\mathrm{Hom}_K(V,W)$ *is* $g \mapsto \chi_V(g^{-1})\chi_W(g)$.

Proof. This is a consequence of (1.8.4), ((2.3)) and ((2.4)). We also prove it by a direct calculation with matrices, thus avoiding (1.8.4). We express the necessary data in matrix form. Let $v_1,\dots,v_m$ be a basis of V and $w_1,\dots,w_n$ a basis of W. We set $l_g^{-1}(v_i) = \sum_j a_{ji}v_j$ and $l_g(w_k) = \sum_l b_{lk}w_l$. Then a basis of $\mathrm{Hom}_K(V,W)$ is $e_{rs}\colon v_i \mapsto \delta_{si}w_r$. We compute:

$$\begin{aligned} & (g\cdot e_{rs})(v_i) = ge_{rs}(g^{-1}v_i) = ge_{rs}(\textstyle\sum_j a_{ji}v_j) = g(\sum_j a_{ji}\delta_{sj}w_r) \\ = \; & \textstyle\sum_{j,l}\delta_{sj}a_{ji}b_{lr}w_l = \sum_l a_{si}b_{lr}w_l = \sum_l a_{si}b_{lr}e_{li}(v_i). \end{aligned}$$

The trace is the sum of the diagonal elements $\sum_{r,s} a_{ss}b_{rr} = \chi_V(g^{-1})\chi_W(g)$. □

We now combine (2.1.1) and (2.1.2) and obtain

$$\langle V,W\rangle = \dim_K \mathrm{Hom}_G(V,W) = |G|^{-1}\textstyle\sum_{g\in G}\chi_V(g^{-1})\chi_W(g). \tag{2.5}$$

This formula tells us that we can compute $\langle V,V\rangle$ from the character. The character does not change under field extensions. We know that $\langle V,V\rangle = 1$ implies that V is irreducible; it then remains irreducible under field extensions. If this is the case, we call the representation ***absolutely irreducible***. When K is algebraically closed, Schur's lemma says $\langle V,V\rangle = 1$. Therefore V is absolutely irreducible if and only if $\langle V,V\rangle = 1$.

(2.1.3) Theorem. *Two representations of G are isomorphic if and only if they have the same character.*

Proof. Let V and V' have the same character. Then, by ((2.5)), the values $\langle W,V\rangle$ and $\langle W,V'\rangle$ are equal for all W. From (1.9.5) we now see that the multiplicities of $W\in \mathrm{Irr}(G,K)$ in V and V' are equal. □

The previous theorem has an interesting consequence; it roughly says, that cyclic subgroups detect representations. If V is a G-representation and H a subgroup of G, we can view V as an H-representation by restriction of the group action. Denote it $\mathrm{res}^G_H V$ for emphasis. The character value $\varphi_V(g)$ only depends on the restriction to the cyclic subgroup generated by g. Therefore:

(2.1.4) Theorem. *G-representations V and W are isomorphic if and only if* $\mathrm{res}^G_H V$ *and* $\mathrm{res}^G_H W$ *are isomorphic for each cyclic subgroup H of G.* □

(2.1.5) Proposition. *Let $V = KS$ be the permutation representation of the finite G-set S. Then $\chi_V(g) = |S^g|$. Here $S^g = \{s \in S \mid gs = s\}$.*

Proof. Consider the matrix of l_g with respect to the basis S. A basis element $s \in S$ yields a non-zero entry on the diagonal if and only if $gs = s$, and this entry is 1. □

(2.1.6) Proposition. *Let K be a splitting field for G and H. Then*

$$\mathrm{Irr}(G;K) \times \mathrm{Irr}(H;K) \to \mathrm{Irr}(G \times H;K), \quad (V,W) \mapsto V \otimes W$$

is a well-defined bijection.

Proof. In the statement of the proposition we view $V \otimes W$ as $G \times H$-representation, as explained in section 1.8. From $\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h)$ and ((2.5)) we obtain

$$\langle V_1 \otimes W_1, V_2 \otimes W_2 \rangle_{G\times H} = \langle V_1, V_2 \rangle_G \langle W_1, W_2 \rangle_H.$$

This shows that $V \otimes W$ is irreducible, if we start with irreducible representations V and W. It also shows that the map in question is injective. We use (1.10.3) and see that we got the right number of irreducible $G \times H$-representations. □

Problems

1. Let $H \lhd G$ and V a G-representation. Then V^H is a G/H-representation. Its character is given by $\chi_{V^H}(gH) = |H|^{-1} \sum_{h\in H} \chi_V(gh)$.

2.2 Orthogonality

We derive orthogonality properties of characters and show that the irreducible characters form an orthonormal basis in the ring of class functions. We assume that K has characteristic zero and is a splitting field for G.

Let $Cl(G,K) = Cl(G)$ be the ring of class functions $G \to K$ (pointwise addition and multiplication). We define on $Cl(G)$ a symmetric bilinear form

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g\in G} \alpha(g^{-1})\beta(g). \tag{2.6}$$

Bilinearity is clear and the reason for symmetry is that we can replace summation over g by summation over g^{-1}. By ((2.5)), $\langle V, W \rangle = \langle \chi_V, \chi_W \rangle$. This gives us together with Schur's lemma the orthogonality properties of characters:

(2.2.1) Proposition. *The irreducible characters form an orthonormal system with respect to the bilinear form ((2.6)).* □

The main result (2.2.5) of this section says that the irreducible characters are a basis of the vector space $Cl(G)$. We prepare for the proof.

(2.2.2) Proposition. *The linear map $q_\alpha = \sum_{g\in G} \alpha(g) l_g \colon V \to V$ is a morphism for each representation V if and only if $\alpha\colon G \to K$ is a class function.*

Proof. Let α be a class function. We compute

$$q_\alpha(hv) = \sum \alpha(g) l_g(hv) = \sum \alpha(g) ghv = \sum \alpha(h^{-1}gh) h(h^{-1}gh) = hq_\alpha(v).$$

For the converse we evaluate the equation $q_\alpha(h) = hq_\alpha(e)$ in the regular representation and compare coefficients. □

(2.2.3) Proposition. *Let α be a class function. Then $p_\alpha = \sum_{g\in G} \alpha(g^{-1}) l_g$ acts on V as the multiplication by the scalar $|G|(\dim V)^{-1}\langle\, \alpha, \chi_V \,\rangle$.*

Proof. By (2.2.2), p_α is an endomorphism of V, and $\langle\, V, V \,\rangle = 1$ tells us that p_α is the multiplication with some scalar λ. The computation (Tr = Trace)

$$\begin{aligned} \lambda \dim V &= \mathrm{Tr}(\lambda \cdot \mathrm{id}) = \mathrm{Tr}\left(\sum \alpha(g^{-1}) l_g\right) \\ &= \sum \alpha(g^{-1}) \,\mathrm{Tr}(l_g) = \sum \alpha(g^{-1}) \chi_V(g) \\ &= |G| \langle\, \alpha, \chi_V \,\rangle. \end{aligned}$$

determines λ. □

(2.2.4) Lemma. *Let $\alpha \in Cl(G)$ be orthogonal to the characters of irreducible representations. Then $\alpha = 0$.*

Proof. The hypothesis of the lemma and (2.2.3) imply that p_α acts as zero morphism in each irreducible representation, hence in each representation. In the regular representation we have $0 = p_\alpha(e) = \sum_g \alpha(g^{-1})g$. Hence $\alpha(g) = 0$ for all $g \in G$. □

(2.2.5) Theorem. *The irreducible characters of G are an orthonormal basis of of $Cl(G)$. The number of irreducible representations is equal to the number of conjugacy classes of G.*

Proof. Let $U \subset Cl(G)$ be a linear subspace. If $U \neq Cl(G)$ then the orthogonal complement $U^\perp$ with respect to $\langle\, -, - \,\rangle$ is different from zero, since $U^\perp$ is the kernel of the linear map $Cl(G) \to \mathrm{Hom}(U, K)$, $x \mapsto (u \mapsto \langle\, x, u \,\rangle)$. For the subspace U generated by characters, $U^\perp = 0$, by (2.2.4), hence $U = Cl(G)$. Now recall (2.2.1).

The dimension of $Cl(G)$ is the number of conjugacy classes, because a basis of $Cl(G)$ consists of those functions which have value 1 on one class and value 0 on all the other classes. □

There exist more general orthogonality relations. They are concerned with the entries of matrix representations and are consequences of the next result.

(2.2.6) Proposition. *Let V be irreducible and suppose that $\langle V, V \rangle = 1$. Then for each linear map $f \in \mathrm{Hom}(V,V)$*

$$\frac{1}{|G|}\sum_{g\in G} l_g f l_g^{-1} = \frac{\mathrm{Tr}(f)}{\dim V}\cdot \mathrm{id}\,.$$

Proof. The left hand side is contained in $\mathrm{Hom}_G(V,V)$ and has the form $\lambda \cdot \mathrm{id}$, since $\langle V, V \rangle = 1$. We apply the trace operator

$$\lambda \dim V = \mathrm{Tr}(\lambda\cdot \mathrm{id}) = |G|^{-1}\textstyle\sum_{g\in G}\mathrm{Tr}(l_g f l_g^{-1}) = |G|^{-1}\sum_{g\in G}\mathrm{Tr}(f) = \mathrm{Tr}(f)$$

and determine λ. □

For $v \in V$ and $\varphi \in V^*$ we obtain from (2.2.6)

$$\textstyle\sum_{g\in G}\varphi(gf(g^{-1}v)) = (\dim V)^{-1}|G|\,\mathrm{Tr}(f)\varphi(v).$$

We apply this to the linear map $f\colon v \mapsto \psi(v)w$, $w \in V$, $\psi \in V^*$ with trace $\mathrm{Tr}(f) = \psi(w)$ and obtain

$$|G|^{-1}\textstyle\sum_{g\in G}\psi(g^{-1}v)\varphi(gw) = (\dim V)^{-1}\varphi(v)\psi(w). \tag{2.7}$$

Note that we can use the definition ((2.6)) of $\langle \alpha, \beta \rangle$ for arbitrary functions $\alpha, \beta\colon G \to K$. This remark can be applied to the left hand side of ((2.7)). Let $v_1, \dots, v_n$ be a basis of V and $\varphi_1, \dots, \varphi_n$ the dual basis. In a matrix representation $gv_i = \sum_j r^V_{ji}(g)v_j$ we have $\varphi_j(gv_i) = r^V_{ji}(g)$. We apply ((2.7)) to this situation and arrive at the following:

(2.2.7) [Orthogonality for matrix entries] Let V and W be irreducible representations of G. Then

$$\langle r^V_{lk}, r^W_{ji} \rangle = \frac{1}{\dim V}\delta_{li}\delta_{jk}\delta_{VW}. \tag{2.8}$$

We have treated the case $V = W$. If V is not isomorphic to W and $f \in \mathrm{Hom}(V,W)$, then the left hand side of the equality in ((2.8)) is zero. □

Problems

1. Let $V_1, \dots V_r$ be a complete set of pairwise non-isomorphic irreducible KG-representations. Let (a^j_{rs}) denote a matrix representation of V_j. Then the functions a^j_{rs} are an orthogonal basis of the space of functions $G \to K$ with respect to the form ((2.6)).

2.3 Complex Representations

We begin with some special and useful properties of representations over the complex numbers.

An (Hermitian) inner product $V \times V \to \mathbb{C}$, $(u,v) \mapsto \langle u, v \rangle$ on a G-representation V is called ***G-invariant*** if $\langle gu, gv \rangle = \langle u, v \rangle$ for $g \in G$ and $u, v \in V$. A representation together with a G-invariant inner product is a ***unitary representation***. A real representation together with a G-invariant inner product is an ***orthogonal representation***.

(2.3.1) Proposition. *A complex representation V of a finite group possesses a G-invariant inner product.*

Proof. Let $b\colon V \times V \to \mathbb{C}$ be any inner product (conjugate-linear in the first variable) and define

$$c(u,v) = \frac{1}{|G|} \sum_{g \in G} b(gu, gv).$$

Then c is linear in v, conjugate linear in u, and G-invariant because of the averaging process. Also it is positive definite and $c(u,v) = \overline{c(v,u)}$. □

Let U be a sub-representation of a unitary representation V. Then the orthogonal complement $U^\perp$ is again a sub-representation an $V = U \oplus U^\perp$. This gives another proof that complex representations are semi-simple. Similarly for orthogonal representations. If we choose an orthonormal basis in an n-dimensional unitary representation, then the associated matrix representation is a homomorphism into the unitary group $G \to U(n)$. In terms of matrix representations, (2.3.1) has the interesting consequence that a homomorphism $G \to GL_n(\mathbb{C})$ of a finite group G is conjugate to a homomorphism $G \to U(n)$.

Let V be a complex representation. There is associated the ***complex-conjugate representation*** $G \times \overline{V} \to \overline{V}$ on the conjugate vector space $\overline{V}$ (the same underlying set and vector addition, but $\lambda \in \mathbb{C}$ now acts as multiplication with $\overline{\lambda}$).

We consider a Hermitian form on V as a bilinear map $\overline{V} \times V \to \mathbb{C}$. Associated is the adjoint $\overline{V} \to V^*$, $v \mapsto (u \mapsto \langle v, u \rangle)$ into the dual vector space. It is an isomorphism of G-representations, in the case of a G-invariant inner product. In terms of characters this means $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. For complex class functions we define a Hermitian form on $Cl(G)$ by

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g). \tag{2.9}$$

The relation $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ shows $(\chi_V, \chi_W) = \langle V, W \rangle$. Therefore the irreducible characters are also an orthonormal basis for this form. Recall the notation $I = \mathrm{Irr}(G; \mathbb{C})$.

Let $C \subset G$ be a representing system for the conjugacy classes. From $|C| = |I|$ we see that $X\colon C \times I \to K,\ (c,V) \mapsto \chi_V(c)$ is a square matrix. It is called the ***character table*** of G.

We express the orthogonality relations in terms of the character table. Let $X^*\colon I \times C \to \mathbb{C}$, $(V,c) \mapsto \overline{\chi_V(c)}$ be the conjugate-transpose and $D\colon C \times C \to \mathbb{C}$ the diagonal matrix $(c,d) \mapsto \delta_{c,d}|c|$, where $|c|$ denotes the cardinality of the conjugacy class of c. The orthogonality relation $(\chi_V, \chi_W) = \delta_{V,W}$ then reads:

(2.3.2) [First orthogonality relation] For irreducible complex representations V and W the relation

$$\sum_{c \in C} |c|\overline{\chi_V(c)}\chi_W(c) = |G|\delta_{V,W}$$

holds. □

In matrix form (2.3.2)says $X^*DX = |G|E$ (unit matrix E). This implies

$$XX^*D = XX^*DXX^{-1} = X(|G|E)X^{-1} = |G|E$$

and then $XX^* = |G|D^{-1}$. Let $Z(c) = \{g \in G \mid gcg^{-1} = c\}$ denote the ***centralizer*** of c in G. Then $|c| = |G/Z(c)|$. We write out the last matrix equation:

(2.3.3) [Second orthogonality relation] For $c, d \in C$ the relation

$$\sum_{V \in I} \overline{\chi_V(c)}\chi_V(d) = \delta_{c,d}|Z(c)|$$

holds. □

(2.3.4) Proposition. *Let $(V, \langle -,- \rangle_V)$ and $(W, \langle -,- \rangle_W)$ be unitary representations. Suppose V and W are isomorphic as complex representations. Then they are isomorphic as unitary representations, i.e., there exists a G-morphism $f\colon V \to W$ such that $\langle f(v_1), f(v_2) \rangle_W = \langle v_1, v_2 \rangle_V$.*

Proof. Let $\varphi\colon V \to W$ be a G-morphism. We use φ to pull $\langle -,- \rangle_W$ back to V, i.e., we define a second inner product $\langle -,- \rangle'$ on V by $\langle v_1, v_2 \rangle' = \langle \varphi(v_1), \varphi(v_2) \rangle_W$. It suffices to produce a G-morphism $\gamma\colon V \to V$ such that $\langle \gamma(v_1), \gamma(v_2) \rangle = \langle v_1, v_2 \rangle'$. We choose an orthonormal basis B of V with respect to $\langle -,- \rangle$ and express everything with respect to this basis. Then $\langle -,- \rangle$ becomes the standard inner product. There exists a positive definite Hermitian matrix A such that $\langle u, v \rangle' = \langle u, Av \rangle = u^t Av$. Since $\langle -,- \rangle'$ is G-invariant, $l_g A = A l_g$. Let $C = \sqrt{A}$ be a positive definite Hermitian matrix. The matrix C also commutes with l_g, since it is a limit of polynomials in A. Then C defines a morphism γ with the desired properties. □

(2.3.5) Corollary. *Let $\alpha, \beta\colon G \to U(n)$ be unitary representations which are conjugate in $GL_n(\mathbb{C})$. Then they are conjugate in $U(n)$. The conjugation matrix can be chosen in $SU(n)$, i.e., to have determinant one.* □

We list a few more properties of complex characters.

(2.3.6) Proposition. *Let χ be the character of a complex representation V. Then:*

(1) $\chi(1) = \dim V$.
(2) $|\chi(g)| \leq \chi(1)$.
(3) $|\chi(g)| = \chi(1)$ *if and only if l_g is the multiplication by a scalar.*
(4) $\chi(g) = \chi(1)$ *if and only if g is contained in the kernel of V.*

Proof. (1) The trace of the identity is $\dim V$.

(2) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of l_g. They are roots of unity, and $\chi(g) = \lambda_1 + \cdots + \lambda_n$. Hence $|\chi(g)| = |\sum \lambda_j| \leq \sum |\lambda_j| = \chi(1)$.

(3) If equality holds, then $\lambda_1 = \cdots = \lambda_n = \lambda$ and l_g is multiplication by λ.

(4) By (3), l_g is the multiplication by 1, if $\chi(g) = \chi(1)$. □

(2.3.7) Remark. If G has a normal subgroup different from 1 and G, then there exists a nontrivial irreducible character χ and $1 \neq g \in G$ such that $\chi(g) = \chi(1)$. Conversely, from (**??**) we see, that if χ and g with these properties exist, then G has a nontrivial normal subgroup. We see that one can obtain group theoretic information from the character table. ◇

The values of complex characters are very special complex numbers. The value $\chi_V(g)$ is the sum of the eigenvalues of l_g, and these eigenvalues are $|g|$-roots of unity ($|g|$ order of g). Let $\mathbb{Z}[\zeta]$ be the subring of the field $\mathbb{Q}(\zeta)$ generated by ζ. The ***exponent*** of a group is the least common multiple of the orders of its elements. Let ζ be a primitive n-root of unity, say $\zeta = \exp(2\pi i/n)$, n the exponent of G. Then χ_V has values in $\mathbb{Z}[\zeta]$. In number theory, the ring $\mathbb{Z}[\zeta]$ is the ring of algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta)$. (An algebraic integer is the root of a monic polynomial with coefficients in $\mathbb{Z}$.)

Problems

1. Express the orthogonality relations (**??**) for complex representations using the Hermitian form ((2.9)).

2. Let V, W be orthogonal representations of G which are isomorphic as real representations. Then they are isomorphic as orthogonal representations.

3. Let V, W be real representations. If they are isomorphic, considered as complex representations, then they are isomorphic as real representations. What does this imply for matrix representations?

4. The character table is a square matrix. Determine the absolute value of its determinant.

2.4 Examples

We study in some detail the groups A_4, S_4, A_5. The symmetric group S_n is the permutation group of $[n] = \{1, \dots, n\}$. The alternating group is the normal subgroup of S_n of even permutations. The geometric significance of the groups in question comes from Euclidean geometry: A_4, S_4, A_5 are the symmetry groups of the tetrahedron, octahedron (cube), icosahedron (dodecahedron), respectively (as far as rotations are concerned , i.e., as subgroups of $SO(3)$).

We begin with some general remarks about permutations. Let $\pi \in S_n$. The cyclic group generated by π acts on $[n]$. We decompose $[n]$ into orbits under this action. An orbit has the form

$$(x, \pi(x), \pi^2(x), \dots, \pi^{t-1}(x)), \qquad \pi^t(x) = x$$

where t is the length of the orbit. We call an orbit a ***cycle*** of the permutation. A permutation can be recovered from its cycles. Therefore we use the cycles to denote the permutation. As an example, the permutation $(318496527) \in S_9$ has the cycles

$$(1,3,8,2),(4),(5,9,7).$$

This means, e.g., that $5 \mapsto 9, 9 \mapsto 7, 7 \mapsto 5$. A cyclic permutation of the entries in a cycle does not change its meaning; thus $(5,9,7) = (9,7,5) = (7,5,9)$. In practice it is not necessary to write cycles of length one, since they just describe fixed points of the permutations. The conceptual significance of the cycles is:

(2.4.1) Proposition. *Permutations in S_n are conjugate elements of the group if and only if for each $k \in \mathbb{N}$ they have the same number of cycles of length k.* □

A ***partition*** of n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ and $\sum_{j=1}^{r} \lambda_j = n$. A conjugacy class of S_n is determined by its associated partition; the λ_j are the lengths of the cycles in the permutation. Thus we have found a combinatorial method to determine the number of irreducible complex representations of S_n; it is the number of partitions of n. The partitions $3, 21, 111$ of 3 tell us that S_3 has 3 irreducible representations.

(2.4.2) Proposition. *Suppose $\pi \in S_n$ has $k(j)$ cycles of length j. The the automorphism group of $[n]_\pi$ has order $1^{k(1)} \cdot k(1)! \cdot 2^{k(2)} \cdot k(2)! \cdot \ldots \cdot n^{k(n)} \cdot k(n)!$. This is the order of the centralizer; hence $n!$, divided by this number, is the size of the conjugacy class of π.* □

(2.4.3) [Representations of S_4] There exist 5 partitions $1111, 211, 22, 31, 4$. We list representing elements of the conjugacy classes and the cardinality of the conjugacy class in the next table. The second row can be obtained from (2.4.2).

1	(12)	(12)(34)	(123)	(1234)
1	6	3	8	6

We turn to the determination of the character table and to the construction of irreducible representations. Names for the five representations and their characters are V_j, $1 \le j \le 5$. We already know 2 one-dimensional representations, the trivial representation V_1 and the sign-representation V_2. Their characters are easily computed.

Character table of S_4					
	1	(12)	(12)(34)	(123)	(1234)
V_1	1	1	1	1	1
V_2	1	-1	1	1	-1
V_3	2	0	2	-1	0
V_4	3	1	-1	0	-1
V_5	3	-1	-1	0	1

We also know already a three-dimensional representation on the space $V_4 = \{(x_1, x_2, x_3, x_4) \mid \sum x_i = 0\}$ by permutation of coordinates. We check again that it is irreducible by computing its character. The character of the permutation representation on $\mathbb{C}^4$ is easily determined by (2.1.5) to have the values $4, 2, 0, 1, 0$. We have to subtract the character of the trivial representation; the result is given in the table. The computation

$$\langle V_4, V_4 \rangle = \tfrac{1}{24} \textstyle\sum_g |\chi_{V_4}(g)|^2 = 3^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot (-1)^2 = 1$$

shows that V_4 is irreducible. The character of $V_5 = V_4 \otimes V_2$ is seen to be as in the table. Thus we found another irreducible representation. We know that the remaining representation must be two-dimensional (1.10.3). It turns out that S_4 has a quotient S_3, the kernel contains (12)(34). We can lift a two-dimensional representation of S_3 to S_4. We lift the analogue of V_4 for S_3. ◇

(2.4.4) [Representations of A_5] We begin again with the determination of the conjugacy classes. We use the cycle notation and have to start with even permutations. But now it is only allowed to conjugate with even permutations, and this has the effect that some of the conjugacy classes of S_5 can split in A_5 into two classes.

(2.4.5) Proposition. *Let $c \in A_n$. Then the S_n-conjugacy class of c is contained in A_n. The S_n-conjugacy class of c split into two A_n-conjugacy classes if and only if the centralizers of c in A_n and S_n coincide. This happens if and only if the partition associated to c consists of different odd numbers.* □

Character table of A_5					
	1	(12)(34)	(123)	(12345)	(13524)
V_1	1	1	1	1	1
V_2	3	-1	0	α	β
V_3	3	-1	0	β	α
V_4	4	0	1	-1	-1
V_5	5	1	-1	0	0

Let ζ be a primitive 5-th root of unity; then $-\alpha = \zeta + \zeta^{-1}$ and $-\beta = \zeta^2 + \zeta^{-2}$. The representation V_1 is trivial. V_4 is the permutation representation on $\{(x_1, \ldots, x_5) \mid \sum x_j = 0\}$. The group A_5 has a subgroup $H \cong D_{10}$. The representation V_5 is obtained from the permutation representation $\mathbb{C}(A_5/H)$ by subtracting the trivial representation. The remaining two representations must be three-dimensional, since $60 - 1^2 - 4^2 - 5^2 = 18 = 3^2 + 3^2$. It is possible to determine the characters without construction of the representations; one uses the fact that the representations cannot have a kernel; and that z^j and z^{-j} are conjugate, so that the character values are real and sums of 5-th roots of unity. The group A_5 has an outer automorphism which interchanges $z = (12345)$ and $z^2 = (13524)$; it is obtained by conjugation with (2354); one verifies $(2354) \circ z \circ (4532) = z^2$. The representation V_3 is obtained from V_2 by this automorphism; and V_2 has a realization over $\mathbb{R}$ as orthogonal symmetry group of the icosahedron. ◇

Problems

1. S_4 acts by conjugation on the set of even permutations of order two. Show that this induces a surjection $S_4 \to S_3$.
2. Compute the number of elements in A_5 of a given order.
3. Determine the irreducible representations and the character table for A_4.
4. Decompose the tensor product of irreducible representations for $G = A_4, S_4, A_5$.
5. Show that S_5 has seven conjugacy classes and irreducible complex representations of dimensions $1, 1, 4, 4, 5, 5, 6$.
6. For a partition $(\lambda_1, \ldots, \lambda_r)$ of n let $S(\lambda) = S(\lambda_1) \times \cdots \times S(\lambda_r)$. Set $V(\lambda) = \mathbb{C}(S_n/S(\lambda))$. Decompose these permutation representations in the cases S_4 and S_5 into irreducibles.

2.5 Real and Complex Representations

Let W be a KG-representation. The involution $T\colon W\otimes W \to W\otimes W, x\otimes y \mapsto y\otimes x$ is a morphism of representations. If K has characteristic different from 2, we split $W\otimes W$ into the ± 1-eigenspaces,

$$S^2(W) = (W\otimes W)_+, \qquad \Lambda^2(W) = (W\otimes W)_-.$$

$S^2(W)$ is the second symmetric power of W and $\Lambda^2(W)$ the second exterior power. Suppose now that $K = \mathbb{C}$. If $w_1,\dots,w_n$ is a basis of W, then $g_{ij} = \frac{1}{2}(w_i\otimes w_j + w_j\otimes w_i), i\le j$ is a basis of $S^2(W)$ and $w_{ij} = \frac{1}{2}(w_i\otimes w_j - w_j\otimes w_i), i<j$ is a basis of $\Lambda^2(W)$. From this information we compute the characters

$$\begin{aligned}\chi_{W\otimes W}(g) &= \chi_W(g)^2\\ \chi_{S^2W}(g) &= \frac{1}{2}(\chi_W(g)^2 + \chi_W(g^2))\\ \chi_{\Lambda^2W}(g) &= \frac{1}{2}(\chi_W(g)^2 - \chi_W(g^2)).\end{aligned}$$

Let W be irreducible. Then $W^* \cong \overline{W}$ is irreducible too. Therefore $\langle W\otimes W, 1\rangle = \langle W, W^*\rangle$ is 1 if $W\cong W^*$ and 0 otherwise. An isomorphism $W\cong\overline{W}$ exists if and only if χ_W is real-valued. In that case we call W ***self-conjugate***. Elements in $\mathrm{Hom}_G(S^2W,\mathbb{C})$ are symmetric G-invariant bilinear forms, and elements in $\mathrm{Hom}_G(\Lambda^2W,\mathbb{C})$ are skew-symmetric G-invariant bilinear forms. Hence W carries a non-zero G-invariant bilinear form if and only if W is self-conjugate. Suppose W is self-conjugate. From $\langle W\otimes W,1\rangle = \langle S^2W,1\rangle + \langle \Lambda^2W,1\rangle$ we see that there two cases: Either $\langle S^2W,1\rangle = 1$, $\langle \Lambda^2W,1\rangle = 0$ or $\langle S^2W,1\rangle = 0, \langle \Lambda^2W,1\rangle = 1$.

(2.5.1) Proposition. *Let W be irreducible. Then*

$$\sigma(W) = \tfrac{1}{|G|}\sum_{g\in G}\chi_W(g^2) = \begin{cases} 0 & W\not\cong W^*\\ 1 & \langle S^2W,1\rangle = 1\\ -1 & \langle \Lambda^2W,1\rangle = 1\end{cases}$$

Proof. By the computation above the sum on the left equals $\langle S^2W,1\rangle - \langle \Lambda^2W,1\rangle$. □

Suppose W is the complexification of a real representation U, i.e., $W\cong \mathbb{C}\otimes U = U_{\mathbb{C}}$. Then U is irreducible and carries a symmetric G-invariant $\mathbb{R}$-bilinear form. This form extends to a symmetric G-invariant $\mathbb{C}$-bilinear form on W. Hence in this case $\sigma(W) = 1$. The converse is also true.

(2.5.2) Proposition. *Suppose $W\in\mathrm{Irr}(G,\mathbb{C})$ carries a G-invariant symmetric form. Then W is the complexification of a real representation.*

Chapter 3

The Group Algebra

3.1 The Theorem of Wedderburn

Assume that the characteristic of K does not divide the order of G. Then the group algebra $A = KG$, considered as a left KG-representation, is semi-simple and hence a direct sum of its isotypical parts. Recall that sub-representations of KG are the same thing as left ideals or as submodules.

(3.1.1) Proposition. *Let $A(V)$ denote the V-isotypical part of A belonging to $V \in \mathrm{Irr}(G; K)$. The $A(V)$ is a two-sided ideal, and every two-sided ideal is a direct sum of isotypical parts.*

Proof. Let $V \subset A$ be irreducible and $a \in A$. Then $W = Va$ is a left ideal, and by Schur's lemma $r_a \colon V \to W, x \mapsto xa$ is either zero or an isomorphism. Hence $W \subset A(V)$.

Let V, W be isomorphic irreducible left ideals. Since A is semi-simple, there exists a projection $f \colon A \to V$. Let $s \colon V \to W$ be an isomorphism. Then $fs(x) = fs(x \cdot 1) = x \cdot sf(1) = x \cdot a$, $a = sf(1)$. If $x \in V$, then $f(x) = x$ and hence $sf(x) = s(x) = xa$, i.e., $W = Va$.

Let $B \subset A$ be a two-sided ideal. Let $V \subset B$ be irreducible and also $W \subset A(V)$. Then $W = Va$. Since B is a right ideal, $W \subset B$ and hence $A(V) \subset B$. □

The isotypical parts are therefore the minimal two-sided ideal. A two-sided ideal is itself an associative algebra, with addition and multiplication inherited from A.

(3.1.2) Proposition. *Let $A = A_1 \oplus \cdots \oplus A_r$ be the decomposition into the minimal two-sided ideal. Then $A_i A_j = 0$ for $i \neq j$.*

Proof. Let $I \subset A_i$ and $J \subset A_j$ be irreducible left ideals. Then $A_i \supset IJ \subset A_j$, since A_i, A_j are two-sided. From $A_i \cap A_j = 0$ we see $IJ = 0$. □

Let $1 = e_1 + \cdots + e_r$, $e_j \in A_j$ be the decomposition of the unit element. Then $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. This follows from

$$\sum e_j = 1 = 1 \cdot 1 = (\sum e_j)(\sum e_k) = \sum e_j e_k = \sum e_j^2.$$

A consequence: e_j is the unit element of the algebra A_j.

Let B be a minimal two-sided ideal of A. A linear subspace V of B is a B-submodule if and only if it is an A-submodule. The map

$$r\colon B \to \mathrm{Hom}_B(B, B), \quad x \mapsto r_x$$

is because of $r_x r_y = r_{yx}$ an anti-isomorphism of algebras. Let $B \cong V^n$, where V is an irreducible submodule of B. Then $\mathrm{Hom}_B(B, B) \cong \mathrm{Hom}_B(V^n, V^n)$. The latter is, by the rules of linear algebra, a matrix algebra. Let $i_j\colon V \to V^n$ be the inclusion of the j-the summand and $p_k\colon V^n \to V$ be the projection onto the k-summand. We associate to $f \in \mathrm{Hom}_B(V^n, V^n)$ the matrix (f_{jk}), $f_{jk} = p_k f i_j \in \mathrm{End}_B(V) = D$. Since V is irreducible, D is a division algebra. Therefore $\mathrm{Hom}_B(V^n, V^n)$ is isomorphic to the matrix algebra $M_n(D)$ of (n, n)-matrices with entries in D. Passage to the transposed matrix is an isomorphism $M_n(D)^\circ \cong M_n(D^\circ)$. (Notation: C° the algebra opposite to C, i.e., order of the multiplication interchanged.) Therefore we have shown in our context:

(3.1.3) Theorem (Theorem of Wedderburn). *The minimal left ideals of the group algebra are isomorphic to matrix-algebras $M_n(D^\circ)$; here $D = \mathrm{End}(V)$ if B is the V-isotypical part, and n is the multiplicity of V in B.* □

If $K = \mathbb{C}$, then the division algebras appearing are just the field $\mathbb{C}$ itself. In the next section we describe the decomposition into matrix algebras in a more explicit manner and relate it to character theory.

3.2 The Structure of the Group Algebra

We assume in this section that K is a splitting field for G of characteristic zero. We write $\dim V = |V|$.

(3.2.1) Proposition. *Let $V \in \mathrm{Irr}(G; K)$. The assignment*

$$t_V\colon \mathrm{Hom}(V, V) \to KG, \quad \alpha \mapsto \tfrac{|V|}{|G|} \textstyle\sum_{g \in G} \mathrm{Tr}(l_g^{-1}\alpha) g$$

is a homomorphism of algebras.

Proof. By comparing coefficients in KG we see that the statement amounts to

$$\tfrac{|V|}{|G|}\sum_{g\in G}\operatorname{Tr}(l_g^{-1}\alpha)\operatorname{Tr}(l_x^{-1}l_g\beta)=\operatorname{Tr}(l_x^{-1}\alpha\beta)$$

for $\alpha,\beta\in\operatorname{Hom}(V,V)$ and $x\in G$. It suffices to prove this for $x=e$.

We use the relation (2.2.6)

$$\tfrac{|V|}{|G|}\sum_{g\in G}l_g\beta l_{g^{-1}}=\operatorname{Tr}(\beta)\operatorname{id}_V,\qquad \beta\in\operatorname{Hom}(V,V)\tag{3.1}$$

and compute

$$\sigma=\tfrac{|V|^2}{|G|^2}\sum_{g,u\in G}l_g\beta l_u l_{g^{-1}}l_{u^{-1}}\alpha$$

in two ways. We apply ((3.1)) to $\sum_u l_u l_{g^{-1}}l_{u^{-1}}$ and use the definition of χ_V; this shows us that σ is equal to

$$\tfrac{|V|}{|G|}\sum_g\chi_V(g^{-1})l_g\beta\alpha.$$

The endomorphism $\frac{|V|}{|G|}\sum_g\chi_V(g^{-1})l_g$ is the identity on V. Hence $\sigma=\beta\alpha$. We now apply ((3.1)) to $\sum_g l_g\beta l_u l_{g^{-1}}$ and obtain

$$\sigma=\tfrac{|V|}{|G|}\sum_u\operatorname{Tr}(\beta l_u)l_{u^{-1}}\alpha.$$

Finally we apply the trace operator to this equation and arrive at

$$\operatorname{Tr}(\beta\alpha)=\tfrac{|V|}{|G|}\sum_u\operatorname{Tr}(l_u\beta)\operatorname{Tr}(l_{u^{-1}}\alpha),$$

and this was to be shown. □

The homomorphism t_V is moreover a morphism of (G,G)-representations, i.e., one verifies directly from the definitions that $t_V(l_g\alpha l_h)=gt_V(\alpha)h$.

(3.2.2) Proposition. *The image of t_V is the V-isotypical part of KG. The (G,G)-representation* $\operatorname{Hom}(V,V)$ *is irreducible and the image of t_V is the* $\operatorname{Hom}(V,V)$*-isotypical part of KG as a (G,G)-representation.*

Proof. The canonical map $V^*\otimes V\to\operatorname{Hom}(V,V)$ is an isomorphism of representations. By (2.1.6), these representations are irreducible. Since t_V is non-zero, t_V is injective. Certainly t_V has an image in the V-isotypical part. We know already that it has dimension $|V|^2=\dim\operatorname{Hom}(V,V)$. Therefore t_V maps isomorphically onto the V-isotypical part. □

The homomorphisms t_V combine to an isomorphism of algebras

$$t\colon\ \bigoplus_{V\in I}\operatorname{Hom}(V,V)\to KG,\quad (x_V)\mapsto\sum_{V\in I}t_V(x_V).$$

This isomorphism induces an isomorphism of the centers of the algebras. The center of $\operatorname{Hom}(V,V)$ consists of the multiples of the identity. Let $Z(A)$ denote the center of the algebra A. We obtain a homomorphism of algebras

$$\tau_V=\operatorname{pr}_V\circ t^{-1}\colon Z(KG)\to Z(\operatorname{Hom}(V,V))\cong\{\lambda\cdot\operatorname{id}\mid\lambda\in K\}\cong K.$$

(3.2.3) Proposition. $\tau_V(\sum_{g\in G}\alpha(g)g) = |V|^{-1}\sum_{g\in G}\alpha(g)\chi_V(g)$.

Proof. The elements $t_V(\mathrm{id}_V) = e_V$ are, by (3.2.2), a vector space basis of $Z(KG)$. Hence it suffices to verify the assertion for these elements. The verification amounts to $\frac{|W|}{|G|}\frac{1}{|V|}\sum_g \chi_W(g^{-1})\chi_V(g) = \delta_{VW}$, and this we know by (2.2.6). □

(3.2.4) Proposition. *The element $\sum_{g\in G}\alpha(g)g \in KG$ is contained in the center of KG if and only if α is a class function.* □

An element $e \in A$ in an algebra A is called ***idempotent***, if it satisfies $e^2 = e$. Idempotents e, f are ***orthogonal***, if $ef = fe = 0$. The elements $e_V \in KG, V \in I$ are pairwise orthogonal, central idempotents. A central idempotent is called ***primitive*** if it is not the sum of two orthogonal (non-zero) idempotents. Since the e_V form a basis of the center of KG, it is easy to verify that the e_V are primitive.

(3.2.5) Proposition. *The multiplication by e_V is in each representation the projection onto the V-isotypical part.* □

(3.2.6) Proposition. *Suppose $V, W \in I(G;\mathbb{C})$. Then the orthogonality relation $\sum_{g\in G}\chi_V(g^{-1})\chi_W(xg) = \frac{|V|}{|G|}\langle V, W\rangle\chi_V(x)$ holds.*

Proof. The relation $e_V e_W = \langle V, W\rangle e_V$ says

$$\frac{|V||W|}{|G|^2}\sum_{g,h}\chi_V(g^{-1})\chi_W(h^{-1})gh = \frac{|V|}{|G|}\langle V, W\rangle\sum_{x\in G}\chi_V(x^{-1})x.$$

Now we compare the coefficients of x^{-1}. □

Chapter 4

Induced Representations

4.1 Basic Definitions and Properties

We compare representations of different groups. The ground field K is fixed.

Let H be a subgroup of G and V an H-representation. Recall the construction $X \times_H Y$ for a right H-set X and a left H-set Y; it is the quotient of the product $X \times Y$ under the equivalence relation $(x, y) \sim (xh^{-1}, hy)$. We denote equivalence classes by their representatives in $X \times Y$. We apply this construction to the right cosets gH, considered as H-sets by right multiplication. We use the bijection $i_g \colon V \to gH \times_H V, v \mapsto (g, v)$ to transport the vector space structure from V to $gH \times_H V$. If we choose another representative $gh \in gH$ of the coset gH, then $i_g l_h = i_{gh}$, and therefore the vector space structure is well-defined. (Although this vector space is just a model of V, we want this model to depend on the coset.) We define a G-action on $\bigoplus_{gH \in G/H} gH \times_H V$; the element $u \in G$ acts as follows

$$gH \times_H V \to ugH \times_H V, \quad (g, v) \mapsto (ug, v).$$

We see that G permutes the summands $gH \times_H V$ transitively. The resulting G-representation is called the ***induced representation*** and is denoted by

$$\mathrm{ind}_H^G V = \bigoplus\nolimits_{gH \in G/H} gH \times_H V. \tag{4.1}$$

(4.1.1) Example. Suppose $|G/H| = 2$. Let $h \mapsto A(h)$ be a matrix representation of H. Fix an element $g \in G \smallsetminus H$. Then a matrix representation for ind_H^G is

$$h \mapsto \begin{pmatrix} A(h) & 0 \\ 0 & A(g^{-1}hg) \end{pmatrix}, \qquad gh \mapsto \begin{pmatrix} 0 & A(ghg) \\ A(h) & 0 \end{pmatrix}.$$

Verify this, using the bijections i_e and i_g. ◇

There exist other constructions of the induced representation, from the view point of set-theory or algebra. Therefore we characterize it by a universal property. The bijection $i_e\colon V \to H \times_H V, v \mapsto (e,v)$ preserves the H-action. We thus obtain an H-morphism

$$i_H^G\colon V \to \operatorname{ind}_H^G V.$$

If W is a G-representation, we denote by $\operatorname{res}_H^G W$ the H-representation obtained from W by restricting the group action to H. The universal property is:

(4.1.2) Proposition. *The assignment*

$$\operatorname{Hom}_G(\operatorname{ind}_H^G V, W) \to \operatorname{Hom}_H(V, \operatorname{res}_H^G W), \quad \Phi \mapsto \Phi \circ i_H^G$$

is a natural isomorphism of vector spaces. In terms of dimensions this implies $\langle \operatorname{ind}_H^G V, W \rangle_G = \langle V, \operatorname{res}_H^G W \rangle_H$.

Proof. From the construction of $\operatorname{ind}_H^G V$ we see that a G-morphism from $\operatorname{ind}_H^G V$ is determined by its restriction to the summand $H \times_H V$. Therefore the map in question is injective.

Conversely, given $\varphi\colon V \to \operatorname{res}_H^G W$, we define a G-morphism $\Phi\colon \operatorname{ind}_H^G V \to W$ on the summand $gH \times_H V$ by $(g,v) \mapsto g \cdot \varphi(v)$. Another representative (gh^{-1}, hv) leads to the same value $gh^{-1}\varphi(hv)$, because φ is an H-morphism. Therefore Φ is well-defined, a G-morphism by construction, and $\Phi \circ i_H^G = \varphi$. □

We refer to (4.1.2) as ***Frobenius reciprocity***. Suppose $j_H^G\colon V \to \tilde{V}$ is an H-morphism into a G-representation $\tilde{V}$ such that $\Phi \mapsto \Phi \circ j_H^G$ induces a bijection $\operatorname{Hom}_G(\tilde{V}, W) \cong \operatorname{Hom}_H(V, \operatorname{res}_H^G W)$. Then there exists a unique isomorphism $\gamma\colon \operatorname{ind}_H^G V \to \tilde{V}$ of G-representations such that $j_H^G = \gamma \circ i_H^G$. This expresses the fact, that (4.1.2) determines the induced representation. One consequence of this fact is the transitivity of induction:

(4.1.3) Proposition. *Let $A \subset B \subset C$ be groups. Then there exists a canonical C-isomorphism $\operatorname{ind}_B^C \operatorname{ind}_A^B V \cong \operatorname{ind}_A^C V$ for A-representations V, since $i_B^C i_A^B$ has the universal property.* □

Given a G-representation, we often ask whether it can be induced from a subgroup H. From the construction of $\operatorname{ind}_H^G V$ we obtain the following answer.

(4.1.4) Proposition. *Let V be an H-sub-representation of the G-representation W. The subspace $gV \subset W$ depends only on the coset gH. We denote it therefore by gHV. Suppose W is the direct sum of the subspaces gHV. Then the canonical map $\operatorname{ind}_H^G V \to W$ associated by* (4.1.3) *to the inclusion $V \subset W$ is an isomorphism.* □

An H-morphism $\alpha\colon V_1 \to V_2$ induces a G-morphism

$$\operatorname{ind}_H^G \alpha\colon \operatorname{ind}_H^G V_1 \to \operatorname{ind}_H^G V_2, \quad (g,v) \mapsto (g, \alpha(v)).$$

In this manner ind_H^G becomes a functor from the category of KH-representations to the category of KG-representations. The isomorphism (4.1.3) is compatible with induced morphisms in the variables V and W. In category theory one says that the induction functor ind_H^G is left adjoint to the restriction functor res_H^G.

Induction preserves direct sums; we have a natural isomorphism

$$\operatorname{ind}_H^G(V_1 \oplus V_2) \cong \operatorname{ind}_H^G V_1 \oplus \operatorname{ind}_H^G V_2. \tag{4.2}$$

Let W be a G-representation. Then the bijections

$$gH \times_H (V \otimes \operatorname{res}_H^G W) \to (gH \times_H V) \otimes W, \quad (g, v \otimes w) \mapsto (g,v) \otimes gv$$

combine to a natural isomorphism of G-representations

$$\operatorname{ind}_H^G(V \otimes \operatorname{res}_H^G W) \cong (\operatorname{ind}_H^G V) \otimes W. \tag{4.3}$$

(4.1.5) Example. Let 1_H denote the trivial one-dimensional H-representation. Then $\operatorname{ind}_H^G 1_H$ is the permutation representation $K(G/H)$. The basis element $gH \in K(G/H)$ corresponds to $(g,1) \in gH \times_H K$.

If V happens to be the restriction of a G-representation $V = \operatorname{res}_H^G W$, then

$$\operatorname{ind}_H^G(V) \cong \operatorname{ind}_H^G(1_H \otimes V) \cong (\operatorname{ind}_H^G 1_H) \otimes W \cong K(G/H) \otimes W.$$

In general one can think of $\operatorname{ind}_H^G V$ as a kind of mixture of the permutation representation $K(G/H)$ with V. ◇

We compute the character of an induced representation in the case that K has characteristic zero.

(4.1.6) Proposition. *Let $W = \operatorname{ind}_H^G V$. Then the character of W is given by the formula*

$$\chi_W(u) = \sum_{gH \in F(u,G/H)} \chi_V(g^{-1}ug) = \frac{1}{|H|} \sum_{g \in C(u,H)} \chi_V(g^{-1}ug)$$

where $C(u,H) = \{g \in G \mid g^{-1}ug \in H\}$ and $F(u,G/H) = G/H^u = \{gH \mid ugH = gH\}$. An empty sum is zero.

Proof. Since $u \in G$ sends $gH \times_H V$ to $ugH \times_H V$, we see that the direct summand $gH \times_H V$ contributes to the trace if and only if $ugH = gH$; and in that case l_u is transformed via the canonical isomorphism i_g into l_h, $h = g^{-1}ug$. □

We define a linear map for class functions by the same formula

$$\operatorname{ind}_H^G\colon Cl(H,K) \to Cl(G,K), \quad (\operatorname{ind}_H^G \alpha)(u) = \frac{1}{|H|} \sum_{g \in C(u,H)} \alpha(g^{-1}ug).$$

We leave it to the reader to verify the next proposition. We use the standard bilinear form ((2.6)) on class functions. The restriction of $\beta \in Cl(G)$ to H is given by composition with $H \subset G$.

(4.1.7) Proposition. *Class functions have the following properties:* $\langle \operatorname{ind}_H^G \alpha, \beta \rangle_G = \langle \alpha, \operatorname{res}_H^G \beta \rangle_H$, $\operatorname{ind}_H^G(\alpha \cdot \operatorname{res}_H^G \beta) = (\operatorname{ind}_H^G \alpha) \cdot \beta$, $\operatorname{ind}_B^C \operatorname{ind}_A^B = \operatorname{ind}_A^C$. □

Problems

1. The character of $\operatorname{ind}_H^G V$ assumes the value 0 at g if g is not conjugate to an element of H.
2. Verify directly that the assignment in (4.1.1) is a homomorphism and that different choices of g lead to conjugate matrix representations.
3. Apply (4.1.1) to the dihedral and quaternion groups D_{2n} and Q_{4n} and compare the result with our earlier constructions.
4. Verify (4.1.7). Verify that $\operatorname{ind}_H^G \alpha$ is a class function.
5. Here is another dual construction of the induced representation, that is also called ***coinduction***. The vector space $\operatorname{Map}_H(G,V)$ of H-equivariant maps $G \to V$ carries a G-action $(u \cdot \varphi)(g) = \varphi(gu)$. The decomposition of G into H-orbits, $G = \amalg Hg$, shows $\operatorname{Map}_H(G,V) \cong \bigoplus_{Hg} \operatorname{Map}_H(Hg,V)$. The assignment

$$\alpha\colon \operatorname{Map}_H(G,V) \to \bigoplus gH \times_H V, \quad \varphi \mapsto \textstyle\sum_{gH}(g, \varphi(g^{-1}))$$

is an isomorphism of G-representations; it sends $\operatorname{Map}_H(Hg,V)$ to $g^{-1}H \times_H V$.
6. The induced representation has, of course, a description in terms of modules. The regular representation KG is a left KG-module and a right KH-module. Let V be a left KH-module. Then the tensor product $KG \otimes_{KH} V$ is a left KG-module. Relate this definition to our first definition of the induced representation; in particular explain from this view point the direct sum decomposition ((4.1))of the induced representation.
7. Let A and B be groups. An (A,B)-set S is a set S with a left A-action and a right B-action which commute $(as)b = a(sb)$, $(a,b)\ inA \times B, s \in S$. Given a finite (A,B)-set S we associate to an A-representation V the vector space $\operatorname{Map}_A(S,V)$ of A-equivariant maps $\varphi\colon S \to V$. This vector space carries a B-action $(b \cdot \varphi)(s) = \varphi(sb)$. A morphism $\alpha\colon V \to W$ of A-representations yields a morphism $\operatorname{Map}_A(S,V) \to \operatorname{Map}_A(S,W)$, $\varphi \mapsto \alpha \circ \varphi$. Let A-Rep denote the category of finite-dimensional left A-representations (over K). The construction above yields a functor

$$\rho(S)\colon A\text{-}\mathrm{Rep} \to B\text{-}\mathrm{Rep}$$

for each finite (A,B)-set S. Let $\gamma\colon S_1 \to S_2$ be a morphism of (A,B)-sets. Composition with γ yields a morphism

$$\rho(\gamma)\colon \operatorname{Map}_A(S_2,V) \to \operatorname{Map}_A(S_1,V),$$

and the family of these morphisms is a natural transformation $\rho(\gamma)\colon \rho(S_2) \to \rho(S_1)$. Altogether we obtain a contravariant functor

$$\rho\colon A\text{-Set-}B \to [A\text{- Rep}, B\text{- Rep}]$$

of the category A-Set-B of finite (A,B)-sets into the functor category.

The composition of functors $\rho(S)$ is again a functor of the same type.

(4.1.8) Proposition. *Let S be an (A,B)-set and T be a (B,C)-set. Then there exists a canonical isomorphism of functors $\rho(T) \circ \rho(S) \cong \rho(S \times_B T)$.*

One has to provide a natural isomorphism

$$\operatorname{Map}_A(S \times_B T, V) \cong \operatorname{Map}_B(T, \operatorname{Map}_A(S,V))$$

of C-representations. It will be induced by the adjunction isomorphism

$$\operatorname{Map}(S \times T, V) \to \operatorname{Map}(T, \operatorname{Map}(S,V)), \quad \varphi \mapsto \hat{\varphi}, \quad \hat{\varphi}(t)(s) = \varphi(s,t).$$

4.2 Restriction to Normal Subgroups

Let H be a subgroup of G. The ***g-conjugate*** ${}^gV$ of the H-representation V is a gHg^{-1}-representation with the same underlying vector space and with action

$$gHg^{-1} \times V \to V, \quad (x,v) \mapsto x \cdot_g v = (g^{-1}xg) \cdot v.$$

The representation ${}^gV$ is irreducible if and only if V is irreducible. For $a,b \in G$ the relation ${}^a({}^bV) = {}^{ab}V$ holds, and g-conjugation is compatible with direct sums and tensor products. For $h \in H$ the map $l_h\colon V \to V$ is an isomorphism ${}^{gh}V \to {}^gV$ of gHg^{-1}-representations. The bijections

$$gH \times_H {}^uV \to guH \times_H V, \quad (g,v) \mapsto (gu,v)$$

combine to an isomorphism of G-representations

$$\operatorname{ind}_H^G {}^uV \cong \operatorname{ind}_H^G V.$$

Now suppose that H is a normal subgroup of G, in symbols $H \lhd G$. Then $gHg^{-1} = H$, and ${}^gV$ is again an H-representation which only depends on the coset gH, up to isomorphism. The group G acts on the set $\operatorname{Irr}(H,K)$

of isomorphism classes of KH-representations by $(g, V) \mapsto {}^gV$. This action factors over an action of G/H since ${}^gV$ only depends on the coset gH. Let

$$G(V) = \{g \in G \mid {}^gV \cong V\}$$

be the isotropy group at V of this G-action; it contains H.

Let W be a G-representation and $V \subset \operatorname{res}^G_H W$ be an H-sub-representation. The left translation $l_g \colon {}^gV \to gV$ satisfies

$$l_g(h \cdot_g v) = g(g^{-1}hgv) = hgv = h \cdot l_g v,$$

and this relation shows that gV is an H-sub-representation which is isomorphic to ${}^gV$ by l_g.

Assume, moreover, that W and V are irreducible. The sum of the gV is a G-sub-representation of W, hence equal to W. Since ${}^gV \cong gV$ is irreducible, $\operatorname{res}^G_H W$ is the sum of irreducible H-representations and therefore semi-simple. Thus we have shown:

(4.2.1) Proposition. *The restriction of a semi-simple G-representation to a normal subgroup H is a semi-simple H-representation.* □

Let again W and $V \subset \operatorname{res}^G_H W$ be irreducible. From our analysis of semi-simple representations we know that $\operatorname{res}^G_H W$ is the direct sum of its isotypical parts, and each irreducible sub-representation of $\operatorname{res}^G_H W$ is isomorphic to some ${}^gV$. Let $W(V)$ be the V-isotypical part of $\operatorname{res}^G_H W$. The $gV, g \in G(V)$ are contained in $W(V)$, and $W(V)$ is the sum of these gV. Therefore $W(V)$ is a $G(V)$-sub-representation of W. The inclusion $W(V) \subset \operatorname{res}^G_{G(V)} W$ gives us, by the universal property (4.1.2) of induced representations, a G-morphism

$$\iota \colon \operatorname{ind}^G_{G(V)} W(V) \to W.$$

In our model of the induced representation, ι maps $gG(V) \times_{G(V)} W(V)$ to $gW(V)$. The subspace $gW(V)$ is another isotypical summand of $\operatorname{res}^G_H W$. From (4.1.4) we obtain:

(4.2.2) Proposition. *ι is an isomorphism of G-representations.* □

Suppose $W(V)$ is isomorphic to r copies of V. Then

$$\operatorname{res}^G_H W \cong r \bigoplus\nolimits_{gG(V) \in G/G(V)} {}^gV.$$

The ${}^gV$, $gG(V) \in G/G(V)$ are pairwise non-isomorphic. The integer r is called the ***ramification index*** of W with respect to the normal subgroup H.

Let $V \in \operatorname{Irr}(H; K)$. The summand $gH \times_H V$ of $\operatorname{ind}^G_H V$ is isomorphic to ${}^gV$; the assignment ${}^gV \to gH \times_H V, v \mapsto (g, v)$ is an isomorphism of H-representations. This shows:

$$\operatorname{res}^G_H \operatorname{ind}^G_H V \cong |G(V)/H| \bigoplus {}^gV. \tag{4.4}$$

The summation is over $gG(V) \in G/G(V)$. We apply Frobenius reciprocity and Schur's lemma to ((4.4)) and obtain

$$\langle \operatorname{ind}_H^G V, \operatorname{ind}_H^G V \rangle_G = |G(V)/H| \sum \langle V, {}^g V \rangle_H = |G(V)/H| \langle V, V \rangle_H. \tag{4.5}$$

From this relation we deduce:

(4.2.3) Proposition. *Let K be algebraically closed of characteristic not dividing $|G|$. Then $\operatorname{ind}_H^G V$ is irreducible if and only if $G(V) = H$.*

Proof. $\operatorname{ind}_H^G V = W$ is irreducible if and only if $\langle W, W \rangle_G = 1$, and this is, by (4.5), the case if and only if $|G(V)/H| = 1$ and $\langle V, V \rangle_H = 1$. □

(4.2.4) Theorem. *Let K be algebraically closed of characteristic zero. Let $H \lhd G$ and $V \in \operatorname{Irr}(H; K)$. Suppose $\operatorname{ind}_H^{G(V)} V = \bigoplus_{j=1}^k m_j V_j$ with pairwise non-isomorphic $G(V)$-representations V_j. Then:*

(1) *$\operatorname{ind}_{G(V)}^G V_j = W_j$ is irreducible.*
(2) *Let $W \in \operatorname{Irr}(G; K)$ and $\langle V, \operatorname{res}_H^G W \rangle \neq 0$. Then $W \cong W_j$ for some j.*
(3) *The W_j are pairwise non-isomorphic.*
(4) *m_j is the ramification index of W_j with respect to H.*
(5) *Let $I^G(V) = \{W_1, \ldots, W_r\} \subset \operatorname{Irr}(G; K)$. Then $I^G({}^g V) = I^G(V)$ and $\operatorname{Irr}(G; K)$ is the disjoint union of the sets $I^G(V)$ where V runs through a representative system of conjugation orbits $\operatorname{Irr}(H; K)/G$.*

Proof. Restriction to H and ((4.4)) yields

$$\bigoplus_{j=1}^k m_j \operatorname{res}_H^{G(V)} V_j = \operatorname{res}_H^{G(V)} \operatorname{ind}_H^{G(V)} V = |G(V)/H| V. \tag{4.6}$$

Therefore $\operatorname{res}_H^{G(V)} V_j = n_j V$ for some $n_j \in \mathbb{N}$. Frobenius reciprocity yields

$$\begin{aligned} n_j = \langle n_j V, V \rangle_H &= \langle V, \operatorname{res}_H^{G(V)} V_j \rangle_H \\ &= \langle \operatorname{ind}_H^{G(V)} V, V_j \rangle_{G(V)} \\ &= m_j \langle V_j, V_j \rangle_{G(V)} = m_j. \end{aligned}$$

Therefore $\operatorname{res}_H^{G(V)} V_j = m_j V$, and together with ((4.6)) we obtain

$$|G(V)/H| = \textstyle\sum_{j=1}^k m_j^2. \tag{4.7}$$

Let now $W \in \operatorname{Irr}(G, K)$ be such that

$$\langle \operatorname{ind}_{G(V)}^G V_i, W \rangle_G \not= 0. \tag{4.8}$$

We want to show that $W \cong W_i = \operatorname{ind}_{G(V)}^G V_i$; this shows in particular that W_i is irreducible, since there exist W such that ((4.8)) holds. By Frobenius reciprocity $0 \neq \langle \operatorname{ind}_{G(V)}^{G} V_i, W \rangle_G = \langle V_i, \operatorname{res}_{G(V)}^{G} W \rangle_{G(V)}$. Therefore V_i occurs in $\operatorname{res}_{G(V)}^{G} W$ and hence $\langle U, V_i \rangle \leq \langle U, \operatorname{res}_{G(V)}^{G} W \rangle$ for each $G(V)$-representation U. In particular

$$\begin{aligned} \langle V, \operatorname{res}_H^G W \rangle &= \langle \operatorname{ind}_H^{G(V)} V, \operatorname{res}_{G(V)}^G W \rangle \\ &\geq \langle \operatorname{ind}_H^{G(V)} V, V_i \rangle = \langle V, \operatorname{res}_H^{G(V)} V_i \rangle = m_i. \end{aligned}$$

This implies, for each $g \in G$,

$$\langle \operatorname{res}_H^G W, {}^g V \rangle = \langle \operatorname{res}_H^G ({}^{g^{-1}} W), V \rangle = \langle \operatorname{res}_H^G W, V \rangle \geq m_i.$$

Therefore each ${}^g V$ occurs in $\operatorname{res}_H^G V$ at least with multiplicity m_i. From ((4.8)) we obtain

$$\dim W \leq \dim \operatorname{ind}_{G(V)}^G V_i. \tag{4.9}$$

Since ${}^g V$ occurs in $\operatorname{res}_H^G V$ at least with multiplicity m_i and since there exist $|G/G(V)|$ different ${}^g V$, we see

$$\dim W \geq m_i |G/G(V)| \dim V = |G/G(V)| \dim V_i = \dim \operatorname{ind}_{G(V)}^G V_i. \tag{4.10}$$

From ((4.9)) and ((4.10)) we obtain equality of dimensions and therefore $W \cong \operatorname{ind}_{G(V)}^G V_i$, since W occurs in $\operatorname{ind}_{G(V)}^G V_i$. Frobenius reciprocity again yields

$$\langle V, \operatorname{res}_H^G W \rangle = \langle \operatorname{ind}_H^G V, W \rangle = \textstyle\sum_j m_j \langle \operatorname{ind}_{G(V)}^G V_j, W \rangle.$$

Therefore $W \in \operatorname{Irr}(G, K)$ is of the form $\operatorname{ind}_{G(V)}^G V_j$ for some $j \in \{1, \dots, k\}$ if and only if $\langle V, \operatorname{res}_H^G W \rangle \neq 0$. There are exactly k G-representations of this type if we show $W_i \ncong W_j$ for $i \neq j$. Suppose that $W_1 \cong W_2$. Then

$$\operatorname{ind}_H^G V = (m_1 + m_2) \operatorname{ind}_{G(V)}^G V_2 + m_3 \operatorname{ind}_{G(V)}^G V_3 + \cdots \tag{4.11}$$

and, together with ((4.7)) and (4.2.3), we arrive at the contradiction

$$\textstyle\sum_j m_j^2 = |G(V)/H| = \langle \operatorname{ind}_H^G V, \operatorname{ind}_H^G V \rangle \geq (m_1 + m_2)^2 + m_3^2 + \cdots + m_k^2;$$

the inequality $\geq$ is a consequence of ((4.11)) and $\langle W_i, W_i \rangle = 1$.

The ramification index of W_j with respect to H is $\langle V, \operatorname{res}_H^G W_j \rangle$. By Frobenius reciprocity this is equal to $\langle \operatorname{ind}_H^G V, W_j \rangle = \langle \sum_t m_t W_t, W_j \rangle = m_j$.

Suppose $W \in I^G(V_1) \cap I^G(V_2)$. Then

$$0 \neq \langle \operatorname{ind}_H^G V_1, \operatorname{ind}_H^G V_2 \rangle = \langle V_1, \operatorname{res} \operatorname{ind} V_2 \rangle.$$

Since $\operatorname{res}_H^G \operatorname{ind}_H^G V_2$ contains only conjugates of V_2, we see that V_1 and V_2 are conjugate. Part (2) and $I^G(V) = I^G({}^g V)$ now shows that $\operatorname{Irr}(G; K)$ is the disjoint union of the $I^G(V)$ as stated. □

(4.2.5) Remark. Theorem (4.2.4) gives a kind of recipe for the construction of irreducible G-representations starting from the irreducible representations of a normal subgroup H.

The situation is easy to survey if V happens to be a restriction of a $G(V)$-representation $\tilde{V}$. In that case $\mathrm{ind}_H^{G(V)} V = \mathrm{ind}_H^G 1_H \otimes \tilde{V}$, see (**??**). Since $H \lhd G(V)$, the representation $\mathrm{ind}_H^{G(V)} 1_H$ is obtained from the regular representation $K(G(V)/H)$ by composition with the quotient homomorphism $G(V) \to G(V)/H$. The decomposition of the regular representation now determines the V_j in (4.2.4). ◇

The next proposition gives conditions under which the lifting property holds for all irreducible representations of H. For the proof go back to (1.6.1).

(4.2.6) Proposition. *Let K be algebraically closed. Suppose H is an abelian normal subgroup and G the semi-direct product of H and P, i.e. $G = HP$ and $H \cap P = 1$. Then each irreducible H-representation V has an extension to $G(V)$.* □

(4.2.7) Remark. We now combine (4.2.4) - (4.2.6). The hypotheses are as in (4.2.6). The irreducible representations of G are obtained as follows. Start with $\eta \in \mathrm{Irr}(H)$. Let $P_\eta \le P$ be the isotropy group of η under the conjugation action of P on $\mathrm{Irr}(H)$. Extend η to $\tilde{\eta}$ by (4.2.6). Let $U \in \mathrm{Irr}(P_\eta)$ and lift to a representation $\tilde{U}$ of HP_η. Then form $W = \mathrm{ind}_{HP_\eta}^G(\tilde{\eta} \otimes \tilde{U})$. The isomorphism class of W uniquely determines the P-orbit of η and the isomorphism class of the P_η-representation U. ◇

4.3 Monomial Groups

The induced representation of a one-dimensional representation is called a ***monomial representation***. A group is called ***monomial*** if each $V \in \mathrm{Irr}(G;\mathbb{C})$ is monomial.

Let $\rho\colon H \to K^*$ be a one-dimensional representation. A basis of $\mathrm{ind}_H^G \rho$ consists of the $(g_j, 1) = x_j$ where $1 = g_1, \dots, g_r$ is a representative system of G/H. Suppose $gg_j = g_{\sigma(j)}h_j$ with $\sigma \in S_r$ and $h_j \in H$. The computation

$$gx_j = (gg_j, 1) = (g_{\sigma(j)}h_j, 1) = (g_{\sigma(j)}, \rho(h_j)) = \rho(h_j)x_{\sigma(j)}$$

shows: The matrix representation of $\mathrm{ind}_H^G \rho$ with respect to the basis above consists of matrices which have in each row and column exactly one non-zero entry. Matrices of this type are called ***monomial***.

A group G is said to be ***supersolvable*** if there exists a string of normal subgroups $1 = G_0 < G_1 < \dots < G_r = G$ such hat G_j/G_{j-1} is a group of prime order. Subgroups and factor groups of supersolvable groups are supersolvable. Groups of prime power order are supersolvable. If H is cyclic and G/H supersolvable, then G is supersolvable.

(4.3.1) Theorem. *Supersolvable groups are monomial.*

The proof needs some preparation and will be finished after (4.3.4).

(4.3.2) Proposition. *Suppose G has an abelian normal subgroup A which is not central. Then a faithful irreducible $\mathbb{C}G$-representation is induced from a proper subgroup.*

Proof. Let $W \in \mathrm{Irr}(G;\mathbb{C})$ be faithful and suppose $V \in \mathrm{Irr}(H;\mathbb{C})$ is an H-sub-representation of W. It suffices to show $G(V) \neq G$; see (4.2.2). Suppose $G(V) = G$. Then $\mathrm{res}^G_A W$ is a multiple of the one-dimensional representation V. Therefore each $a \in A$ acts on V as multiplication by a scalar, hence l_a commutes with l_g for each $g \in G$. Since W is faithful this fact implies that a is contained in the center of G. □

(4.3.3) Lemma. *Let G be a non-abelian supersolvable group. Then G contains a non-central normal abelian subgroup.*

Proof. Let $Z < G$ be the center of G. Since G is supersolvable, so is G/Z. Let $1 \neq H/Z \lhd G/Z$ be a cyclic normal subgroup. Then H is an abelian non-central normal subgroup. □

We need a formal property of induced representations. Let $\alpha\colon A \to B$ be a homomorphism. We associate to a KB-representation V a KA representation $\alpha^* V$ with the same underlying vector space V and with action

$$A \times V \to V, \quad (a,v) \mapsto \alpha(a) \cdot v.$$

In the case that $\alpha\colon A \subset B$ we have $\alpha^* V = \mathrm{res}^B_A V$. If α is surjective, we say that $\alpha^* V$ is obtained from V by ***lifting*** the group action along α. We show that induction is compatible with lifting. Consider

$$\begin{array}{ccc} \tilde\alpha^{-1}(A) = P & \xrightarrow{\ \subset\ } & Q \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde\alpha} \\ A & \xrightarrow{\ \subset\ } & B \end{array}$$

$\tilde\alpha$ surjective and $\alpha = \tilde\alpha | P$. Then

(4.3.4) Lemma. $\tilde\alpha^*(\mathrm{ind}^B_A V) \cong \mathrm{ind}^Q_P(\alpha^* V)$ *for each A-representation V.*

Proof. α induces a bijection $Q/P \to B/A$ by passing to quotients. The isomorphism is induced by $qP \times_P \alpha^* V \to \alpha(q)A \times_A V$, $(q,v) \mapsto (\alpha(q), v)$. □

Proof. (Of (4.3.1).) Let W be a faithful irreducible $\mathbb{C}G$-representation. If G is abelian, then $\dim W = 1$, and nothing is to prove. Otherwise W is, by (4.3.2) and (4.3.3), induced from a proper subgroup $W = \mathrm{ind}_H^G V$. By induction we can assume that V is monomial, and by transitivity of induction, W is monomial.

If W is not faithful, let L be its kernel, and consider W as G/L-representation $\overline{W}$. By induction again, $\overline{W}$ is monomial. Now use (4.3.4). □

Problems

1. Consider the semi-direct product G of the quaternion group $Q_8 = \langle\, a, b \mid a^2 = b^2, bab^{-1} = a^{-1}\,\rangle$ by $C_3 = \langle\, h\,\rangle$ with respect to the automorphism $hah^{-1} = b, hbh^{-1} = ab$. The faithful representation $G \to SU(2)$

$$a \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad h \mapsto \frac{i-1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

is not monomial: The group G has no subgroup of index 2. The group G has the quotient A_4 and is solvable. (G is called the ***binary tetrahedral group***.)

4.4 The Character Ring and the Representation Ring

Let K be a field of characteristic zero. Recall that the characters of the irreducible KG-representations are linearly independent in the ring of class functions $Cl(G;K)$. The additive subgroup $CH(G;K)$ of $Cl(G;K)$ generated by the characters of irreducible representations is therefore a free abelian group of rank $|\operatorname{Irr}(G;K)|$. The relation $\chi_{V\oplus W} = \chi_V + \chi_W$ shows that each character is contained in this group. And the relation $\chi_{V\otimes W} = \chi_V\chi_W$ is used to show that $CH(G;K)$ is a subring of $Cl(G;K)$. This ring is called the ***character ring*** of KG-representations.

The character ring can be constructed formally. It is then called the ***representation ring*** or ***Green ring***. In this context K can be an arbitrary field. Let $R(G;K)^+$ denote the set of isomorphism classes of KG-representations. Direct sum and tensor product induces on $R(G;K)^+$ two composition laws (addition and multiplication), and with these structures $R(G;K)^+$ is almost a commutative ring, except that inverses for the additive structure are missing. In situations like this, there exists a universal ring $R(G;K)$ together with a homomorphism

$$\iota\colon R(G;K)^+ \to R(G;K)$$

of semi-rings which is determined, up to a unique isomorphism, by a universal property: Let $\varphi\colon R(G;K)^+ \to A$ be a homomorphism into an abelian group.

Then there exists a unique homomorphism of abelian groups $\Phi\colon R(G;K) \to A$ such that $\Phi \circ \iota = \varphi$. If, moreover, A is a commutative ring and φ a homomorphism of semi-rings, then the universal homomorphism Φ is a homomorphism of rings. The universal property is used to show that additive constructions with representations extend to the representation ring. Elements in $R(G;K)$ are formal differences $[V]-[W]$ of representations V, W, called ***virtual representations*** and $[V]-[W] = [V']-[W']$ if and only if $V \oplus W' \oplus Z \cong V' \oplus W \oplus Z$ for some representation Z.

(4.4.1) Proposition. *If K has characteristic zero, then the homomorphism $R(G;K)^+ \to CH(G;K)$, $V \mapsto \chi_V$ is a model for the universal ring. If the characteristic of K does not divide the order of the group, then $R(G;K)$ has a $\mathbb{Z}$-basis of isomorphism classes of irreducible representations.* □

Typical additive constructions are restriction and induction.

(4.4.2) Proposition. *Suppose $H \leq G$. Restriction induces a ring homomorphism $\mathrm{res}^G_H\colon R(G;K) \to R(H;K)$. Induction induces an additive homomorphism $\mathrm{ind}^G_H\colon R(H;K) \to R(G;K)$. The relation ((4.3)) shows $\mathrm{ind}^G_H(x \cdot \mathrm{res}^G_H\, y) = (\mathrm{ind}^G_H\, x) \cdot y$. It implies that the image of ind^G_H is an ideal.* □

(4.4.3) Example. Let $G = C_m = \langle\, a \mid a^m = 1\,\rangle$ and $\rho\colon a \mapsto \exp(2\pi i/m)$ the standard representation. Then $R(G;\mathbb{C})$ is the free abelian group with basis $1, \rho, \rho^2, \dots, \rho^{m-1}$. The multiplicative properties of the ρ^k, namely $\rho^k \otimes \rho^l \cong \rho^{k+l}$, show that the ring $R(G;\mathbb{C})$ is isomorphic to $\mathbb{Z}[\rho]/(\rho^m - 1)$. More formally: For a finite abelian group G with character group G^* the representation ring $R(G;\mathbb{C})$ is isomorphic to the group ring $\mathbb{Z}[G^*]$. ◇

(4.4.4) Example. We determine $R(C_m;\mathbb{Q})$ for $C_m = \langle x \mid x^m = 1 \rangle$. Decompose $x^m - 1 \in \mathbb{Q}[x]$ into irreducible factors $x^m - 1 = \prod_{d|m} \Phi_d(x)$. The cyclotomic polynomial $\Phi_d(x)$ has the primitive d-th roots of unity as its roots. The quotient $V_d = \mathbb{Q}[x]/(\Phi_d(x)$, viewed as a module over the group ring $\mathbb{Q}C_m = \mathbb{Q}[x]/(x^m - 1)$, is an irreducible $\mathbb{Q}C_m$-representation. The $V_d, d|m$ form a $\mathbb{Z}$-basis of $R(C_m, \mathbb{Q})$. There is another $\mathbb{Z}$-basis which consists of the permutation representations $\mathbb{Q}(C_m/C_n) = P_{m/n}$. The representation $P_{m/n}$ contains the irreducible representations which have C_n in its kernel. The kernel of V_d is $C_{m/d}$. Hence $P_k = \sum_{d|k} V_d$. By Möbius-inversion one obtains $V_k = \sum_{d|k} \mu(k/d) P_d$. ◇

(4.4.5) Example. Suppose the characteristic of K does not divide the order of G. We describe $R(C_m;K)$. Let $L = K(\varepsilon)$ be the field extension, ε a primitive m-th root of unity. Then, as in (4.4.3), $R(C_m;L) \cong \mathbb{Z}[\rho]/(\rho^m - 1)$ where $\rho\colon C_m \to L^*$ is given by $\rho(a) = \varepsilon$. Field extension yields an injective homomorphism $\iota\colon R(C_m;K) \to R(C_m;L)$. Let $\Gamma = \mathrm{Gal}(L|K)$ be the Galois group of L over K. An element $\gamma \in \Gamma$ is determined by its value $\gamma(\varepsilon) = \varepsilon^t$; and

t is determined modulo m; hence $\gamma \mapsto t$ yields an injection $\Gamma \subset \mathbb{Z}/m^*$. The group Γ acts on $\{1, \varepsilon, \ldots, \varepsilon^{m-1}\}$. If C is an orbit, then $q_C = \prod_{\alpha \in C}(X - \alpha)$ is an irreducible factor of $x^m - 1 \in K[x]$. These irreducible factors correspond to the irreducible KC_m-representations. The group Γ acts on $\mathrm{Irr}(G; L) = \{\rho^t \mid t \in \mathbb{Z}/m\}$ by $(\gamma\rho)(a) = \gamma(\rho(a))$, and hence on $R(G; L)$ by ring automorphisms. The homomorphism ι induces an isomorphism $\iota\colon R(C_m; K) \cong R(C_m; L)^\Gamma$ with the Γ-fixed subring. $\diamond$

Problems

1. Compute $R(D_{2n}; \mathbb{C})$ and study the restriction to $R(C_n; \mathbb{C})$. Compare $R(D_{2n}; \mathbb{R})$ via complexification with $R(D_{2n}; \mathbb{C})$.
2. Let $x \in R(G; \mathbb{C})$ be a unit of finite order. Then the character values of x are roots of unity. This implies that $\langle x, x \rangle = 1$. One concludes that $x = \pm\chi$, where χ is a one-dimensional representation.
3. The group ring $\mathbb{Z}G$ of a finite abelian group G is isomorphic to the representation ring $R(G; \mathbb{C})$. Hence the units of finite order in $\mathbb{Z}G$ are precisely the elements $\pm g$ for $g \in G$. (**??**)
4. The complexifications of representations induces an injective homomorphism $c\colon R(G; \mathbb{R})^* \to R(G; \mathbb{C})^*$. If $x \in R(G; \mathbb{R})$ is a positive unit of finite order (positive: $x(1) > 0$), then $c(x)$ is a one-dimensional character with real values, hence a homomorphism $G \to \mathbb{Z}^* = \{\pm 1\}$. Hence: $\mathrm{Hom}(G, \mathbb{Z}^*)$ is canonically isomorphic to the group of positive units of finite order in $R(G; \mathbb{R})$. Since these units are rational representations, $R(G; \mathbb{Q})$ has the same units of finite order as $R(G; \mathbb{R})$.

4.5 Cyclic Induction

Let $\mathcal{F}$ be a set of subgroups of G. A general question of induction theory is: For which sets $\mathcal{F}$ is the induction map

$$i_{\mathcal{F}} = \langle \mathrm{ind}_H^G \mid H \in \mathcal{F} \rangle\colon \bigoplus_{H \in \mathcal{F}} R(H; K) \to R(G; K)$$

surjective? We know that the image of $i_{\mathcal{F}}$ is an ideal. Therefore $i_{\mathcal{F}}$ is surjective if and only if the unit element 1_G of $R(G; K)$ is contained in the image of $i_{\mathcal{F}}$. It is also interesting to look for integral multiples of 1_G in the image of $i_{\mathcal{F}}$.

Let K be a field of characteristic zero. In this section we prove Artin's induction theorem which says that $|G| 1_G$ is in the image of $i_{\mathcal{C}}$ for the set $\mathcal{C}$ of cyclic subgroups. The proof is based on a character calculation.

We rewrite the basic orthogonality relation

$$|G| \dim V^G = \sum_{g \in G} \chi_V(g). \tag{4.12}$$

Let $C^{\#}$ denote the set of generators of the cyclic group C. Since each element generates a unique cyclic subgroup we can write the right hand side of ((4.12)) as a sum over the cyclic subgroups C of G

$$|G| \dim V^G = \sum_C (\sum_{g \in C^{\#}} \chi_V(g)). \qquad (4.13)$$

We apply this to the cyclic subgroup C itself.

Let $\mu\colon \mathbb{N} \to \mathbb{Z}$ be the Möbius-function, defined inductively by[1] $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for $n > 1$. Let f and g be functions from $\mathbb{N}$ into some (additive) abelian group such that $f(n) = \sum_{d|n} g(d)$; then ***Möbius inversion*** tells us that $g(n) = \sum_{d|n} \mu(n/d) f(d)$.

Note that for each divisor d of $|C| = n$ there exists a unique subgroup $D \le C$ with $|D| = d$. We obtain by Möbius inversion from ((4.13))

$$\sum_{c \in C^{\#}} \chi_V(c) = \sum_{D \le C} \mu(|C/D|)(\sum_{d \in D} \chi_V(d)). \qquad (4.14)$$

The inner sum in ((4.14)) equals $|D| \dim V^D$. Therefore we obtain altogether:

$$|G| \dim V^G = \sum_C (\sum_{D \le C} \mu(|C/D|)|D| \dim V^D). \qquad (4.15)$$

This equality has the form $|G| \dim V^G = \sum_C a_C \dim V^C$ with suitable integers a_C. We use $\langle K(G/H), V \rangle_G = \dim V^H$ in ((4.15)) and see that $\langle |G|K(G/G) - \sum_C a_C K(G/C), V \rangle_G = 0$ for each representation V. This implies that the left argument of the bracket is zero.

(4.5.1) Proposition. *$|G|[K(G/G)] = \sum_C a_C [K(G/C)]$ in $R(G;K)$. Note that $1_G = [K(G/G)]$ is the unit element in $R(G;K)$.* □

We know that $\mathrm{ind}_C^G \mathrm{res}_C^G\colon R(G;K) \to R(G;K)$ is multiplication by $[K(G/C)]$, see (4.1.5). Hence we obtain from (4.5.1):

(4.5.2) Theorem. *For $x \in R(G;K)$ the identity $\sum_C a_C \mathrm{ind}_C^G \mathrm{res}_C^G x = |G|x$ holds. This implies* **Artin's induction theorem***: $|G|R(G;K)$ is contained in the image of $i_{\mathcal{C}}$.* □

(4.5.3) Theorem. *Let V and W be $\mathbb{Q}G$-representations. Suppose that for each cyclic subgroup $C \subset G$ we have $\dim V^C = \dim W^C$. Then V and W are isomorphic.*

Proof. It suffices to show $\mathrm{res}_C^G V \cong \mathrm{res}_C^G W$ for each cyclic subgroup C of G, since we know that representations are determined by their restriction to cyclic subgroups. From our analysis of irreducible $\mathbb{Q}C$-representations we conclude that they are determined up to isomorphism by fixed point dimensions of subgroups. □

[1] $d|n$ means, d divides n.

(4.5.4) Proposition. *The rank of* $R(G;\mathbb{Q})$ *equals the number of conjugacy classes of cyclic subgroups of* G.

Proof. Since conjugate subgroups have fixed points of equal dimension we conclude from (4.5.3) that the rank is at most the number of cyclic conjugacy classes.

We show that the permutations representations $U_C = \mathbb{Q}(G/C)$, (C) cyclic conjugacy class, are linearly independent. Suppose $\sum_{(C)} a_C U_C = 0$. Let C be maximal such that $a_C \neq 0$, and let $g \in C$ be a generator. The character of $\sum_{(C)} a_C U_C$ at g is $a_C |G/C^C| \neq 0$; a contradiction. □

(4.5.5) Proposition. *Let* $\alpha \in Cl(G;K)$. *Suppose for each cyclic subgroup* C *the restriction* $\mathrm{res}^G_C\,\alpha \in R(C;K)$. *Then* $|G|\alpha \in R(G;K)$.

Proof. This is a direct consequence of (4.5.2). □

Problems

1. Let $G = A_5$. The virtual permutation representation associated to $[G/C_5] + [G/C_3] + [G/C_2] - [G/C_1]$ realizes $2 \cdot 1_G \in R(G;\mathbb{Q})$.

4.6 Induction Theorems

In order to state further induction theorems we have to specify suitable sets of subgroups. Let p be a prime number. A p-group is a group of p-power order. Let $|G| = p^t q$ with $(p,q) = 1$. Then there exists a subgroup $G(p) \leq G$ of order p^t, and all such groups are conjugate; they are called ***Sylow*** ***p-groups*** of G. A ***p-hyperelementary group*** H is the semi-direct product of a cyclic group and a p-group P of coprime order; S is a normal subgroup of H and $H/S \cong P$. Let $\mathcal{H}(p,G)$ denote the set of p-hyperelementary subgroups of G. The set $\mathcal{H}(G) = \cup_p \mathcal{H}(p,G)$ is the set of ***hyperelementary*** subgroups of G. Hyperelementary groups are monomial. A ***p-elementary*** group H is the direct product $S \times P$ of a cyclic group S and a p-group P of coprime order. We denote by $\mathcal{E}(p,G)$ the set of p-elementary subgroups of G and by $\mathcal{E}(G) = \cup_p calE(p,G)$ the set of ***elementary*** subgroups of G.

As in the case of Artin's induction theorem, the hyperelementary induction theorem is a consequence of a result about permutation representations.

(4.6.1) Theorem. *Let* K *be a field of characteristic zero. There exists in* $R(G;K)$ *a relation of the type* $|G/G(p)|1_G = \sum_{E \in \mathcal{H}(p,G)} h_E [K(G/E)]$ *with suitable integers* h_E.

We defer the proof to a later section where we deal systematically with such results; see (**??**).

(4.6.2) Theorem (Hyperelementary induction). *Let K be a field of characteristic zero. Then $|G/G(p)|R(G;K)$ is contained in the image of $i_{\mathcal{H}(p,G)}$. The induction map $i_{\mathcal{H}(G)}$ is surjective.*

Proof. The first assertion is a consequence of (4.6.1). The integers $|G/G(p)|$, p a divisor of $|G|$, have no common divisor. Hence there exist integers n_p such that $\sum_p n_p|G/G(p)| = 1$. We conclude that 1_G is in the image of $i_{\mathcal{H}(G)}$. □

Hyperelementary groups are supersolvable. Therefore the next theorem is a consequence of (4.6.2).

(4.6.3) Theorem (Monomial induction). *$R(G;\mathbb{C})$ is generated by monomial representations, i.e., each element $x \in R(G;\mathbb{C})$ is of the form $x = [V]-[W]$ with representations V and W which are direct sums of monomial representations.* □

(4.6.4) Proposition. *Let $H = SP$ be a p-hyperelementary group. Then $i_{\mathcal{E}(p,H)}$ is surjective ($K = \mathbb{C}$).*

Proof. By induction on $|H|$ we can assume that irreducible representations of dimension greater than one are in the image of the induction map. Let $\alpha \in X(H)$ be a one-dimensional representation. Consider the elementary subgroup $E = S^P \times P$. We claim: If $\gamma \in X(H)$ occurs in $\mathrm{ind}_E^G \mathrm{res}_E^G \alpha$, then $\alpha = \gamma$ and α occurs with multiplicity one. By Frobenius reciprocity this is a consequence of (1.6.3). Thus modulo representations of dimensions greater that one, each element of $X(H)$ is in the image of the induction map. □

We combine (4.6.2) and (4.6.4) and obtain:

(4.6.5) Theorem (Brauer's induction theorem). *Let $K = \mathbb{C}$. Then $|G/G(p)|R(G;\mathbb{C})$ is contained in the image of $i_{\mathcal{E}(p,G)}$, and $i_{\mathcal{E}(G)}$ is surjective.* □

We now derive an interesting consequence of the monomial induction theorem. The ***exponent*** $e(G)$ of a group G is the least common multiple of the order of its elements; it divides $|G|$.

(4.6.6) Theorem (Splitting field). *Let ε be a primitive $e(G)$-th root of unity. Then $\mathbb{Q}(\varepsilon)$ is a splitting field for G, i.e., each irreducible $\mathbb{C}G$-representation has a realization with matrices having entries in $\mathbb{Q}(\varepsilon)$.*

Proof. One-dimensional representations of subgroups of G are certainly realizable over $\mathbb{Q}(\varepsilon)$ and therefore also monomial representations, being induced from one-dimensional ones. From (4.6.3) we infer: Let M be a $\mathbb{C}G$-representation.

Then there exist representations U and V, which are realizable over $\mathbb{Q}(\varepsilon)$ and such that $U \cong V \oplus M$. It is now a general fact that this implies: M is realizable over $\mathbb{Q}(\varepsilon)$. See the next proposition. □

Let L be an extension field of K (of characteristic zero). We denote by $V_L = L \otimes_K V$ the extension of a KG-representation to an LG-representation. From character theory we see $\langle V, W \rangle_{KG} = \langle V_L, W_L \rangle_{LG}$.

(4.6.7) Proposition. *Let U and V be KG-representations and M an LG-representation. Suppose $U_L \cong V_L \oplus M$. Then there exists a KG-representation N such that $N_L \cong M$.*

Proof. We fix M. Among all possible isomorphisms $U_L \cong V_L \oplus M$ choose one with V of smallest dimension. We show $V = 0$. Suppose $V \neq 0$. Choose $W \in \mathrm{Irr}(G; K)$ with $\langle W, V \rangle_{KG} > 0$. Then

$$\begin{aligned}\langle U, W \rangle_{KG} &= \langle U_L, W_L \rangle_{LG} = \langle V_L \oplus M, W_L \rangle_{LG} \\ &= \langle V, W \rangle_{KG} + \langle M, W_L \rangle_{LG} > 0.\end{aligned}$$

Therefore W occurs in U, hence $U \cong U' \oplus W$, $V \cong V' \oplus W$ by semi-simplicity. We conclude $U'_L \oplus W_L \cong V'_L \oplus W_L \oplus M$, cancel W_L, and see that V was not of minimal dimension. □

(4.6.8) Proposition. *The restriction $\rho\colon R(G;\mathbb{C}) \to \prod_E R(G;\mathbb{C})$ is an injection as a direct summand (E elementary subgroups).*

Proof. Suppose $1_G = \sum_E \mathrm{ind}_E^G x_E$. Define

$$\lambda\colon R(G) \to \prod_E R(E), \quad x \mapsto (res_E^G x \cdot x_E \mid E).$$

Then $\sum_E \mathrm{ind}_E^G(res_E^G \cdot x_E) = \sum_E x \cdot \mathrm{ind}_E^G x_E = x \cdot 1_G = x$. Hence λ is a splitting of the induction.

Dually, define

$$r\colon \bigoplus_E R(E) \to R(G), \quad (y_E) \mapsto \sum_E \mathrm{ind}_E^G(y_E \cdot x_E).$$

Then $r\rho(x) = \sum \mathrm{ind}_E^G(\mathrm{res}_E^G x \cdot x_E = x \cdot \sum_E \mathrm{ind}_E^G x_E) = x$. Thus r is a splitting of ρ. □

(4.6.9) Proposition. *Let $\alpha \in Cl(G)$ be such that $\mathrm{res}_E^G \alpha \in R(E)$ for each elementary subgroup E of G. Then $\alpha \in R(G)$.*

Proof. This is a consequence of (4.6.7). □

For arbitrary fields (of characteristic zero) we have results of the type (4.6.7) and (4.6.8), using hyperelementary groups. The proofs are the same.

Problems

1. Verify the elementary induction theorem explicitly for $G = A_5$.
2. The virtual permutation representation associated to $[G/D_5] + [G/D_3] - [G/D_2]$ is the unit element, $G = A_5$.

4.7 Elementary Abelian Groups

We assume that p does not divide the characteristic of K.

(4.7.1) Theorem. *Let V be a faithful KG-representation and $A \lhd G$ an elementary abelian group of rank $n \geq 2$. Then there exists $H \leq A$ such that $|A/H| = p$ and $V^H \neq 0$. The normalizer $N_G H$ is different from G and the canonical map $\mathrm{ind}_{NH}^G V^H \to V$ is an isomorphism.*

The proof needs some preparation. Let $S(A) = \{H \leq A \mid |A/H| = p\}$ be the set of cocyclic subgroups. Consider the following elements in the group algebra KA

$$x_H = |A|^{-1}(p\Sigma_H - \Sigma_A), \qquad y = |A|^{-1}\Sigma_A$$

with $\Sigma_A = \sum_{a \in A}$ and $\Sigma_H = \sum_{h \in H} h$ for $H \in S(A)$.

(4.7.2) Proposition. *The elements x_H and y have the following properties:*
(1) $x_H^2 = x_H$; $y^2 = y$
(2) $x_H x_K = 0$ *for* $H \neq K$; $x_H y = 0$
(3) $1 = y + \sum_{H \in S(A)} x_H$.

Proof. The proof of (1) and (2) is a direct consequence of the relations $\Sigma_A^2 = |A|\Sigma_A$, $\Sigma_H \Sigma_A = |H|\Sigma_A$, $\Sigma_H^2 = \Sigma_H$ and $\Sigma_H \Sigma_K = p^{-2}|A|\Sigma_A$ for $H \neq K$. In order to prove (3), one has to count the number of $H \in S(A)$ which contain $1 \neq a$ and the cardinality of $S(A)$. The latter equals the number $(p^n - 1)/(p-1)$ of one-dimensional subspaces of A. The former is the number of subspaces of $A/\langle\, a\, \rangle$ of codimension one. From this information one verifies (3). □

(4.7.3) Proposition. *Let V be a KA-representation. Then $yV = Y^A$ and $x_H V \oplus V^A = V^H$.*

Proof. We already know that multiplication with y is a projection operator onto the fixed point set. The second assertion follows from $|H|^{-1}\Sigma_H = x_h + y$, $x_H y = 0$, and the fact that multiplication by $|H|^{-1}\Sigma_H$ is the projection onto V^H. □

(4.7.4) Proposition. $V = yV \oplus \bigoplus_{H \in S(A)} x_H V$.

Proof. (4.7.2) (3) shows that V is the sum of yV and the $x_H V$, and (4.7.2) (2) shows that the sum is direct. □

(4.7.5) Corollary. *Let* $S(A,V) = \{H \in S(A) \mid V^H \neq 0\}$ *and suppose* $V^G = 0$. *Then* $V = \bigoplus_{H \in S(A,V)} V^H$. *In particular* $S(A,V) \neq \emptyset$. □

Proof. (Of (4.7.1)). By (4.7.4), there exists $H \in S(A)$ such that $V^H \neq 0$. Since $gV^H = V^{gHg-1}$, the group G acts on $S(A,V)$ by conjugation. Since $\sum gV^H = V$, the action is transitive, and NH is the isotropy group of $H \in S(A,V)$. The statement is now a special case of (4.2.1). □

We report on group to which (4.7.1) applies.

(4.7.6) Theorem. *Suppose each abelian normal subgroup of the p-group G is cyclic. Then G is a group in the following list.*

(1) *G is cyclic.*
(2) *G is the dihedral group $D(2^n)$ of order 2^n, $n \geq 4$. It has the presentation* $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{-1} \rangle$.
(3) *G is the semi-dihedral group $SD(2^n)$ of order 2^n, $n \geq 4$. It has the presentation* $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{2^{n-2}-1} \rangle$.
(4) *G is the quaternion group $Q(2^n)$ of order 2^n, $n \geq 3$. It has the presentation* $\langle A, B \mid B^2 = A^{2^{n-2}}, BAB^{-1} = A^{-1} \rangle$. □

We have shown that complex representations of p-groups are induced from one-dimensional representations. The virtue of (4.7.1) is that we do not need any hypothesis about the field K, except that we are in the semi-simple case. Thus (4.7.1) applies, e.g., to real or rational representations. It then remains to study the groups in the list (4.7.5). The situation is particularly simple for $p \neq 2$, since then everything is reduced to the cyclic groups.